REGULAR SYSTEMS OF EQUATIONS IN $\lambda$-CALCULUS

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ABSTRACT

Many problems arising in equational theories like Lambda-calculus and Combinatory Logic can be expressed by combinatorial equations or systems of equations. However, the solvability problem for an arbitrarily given class of systems is in general undecidable. In this paper we shall focus our attention on a decidable class of systems, which will be called regular systems, and we shall analyse some classical problems and well-known properties of Lambda-calculus that can be described and solved by means of regular systems. The significance of such class will be emphasized showing that for slight extensions of it the solvability problem turns out to be undecidable.

Keywords: Combinatory equations and systems of equations; ($X$)-separability, invertibility and strong normalization in $\lambda$-calculus; Fixed point of combinators; Numerical systems; Functional programming.

0. Introduction

Many problems arising in equational theories like $\lambda$-calculus and Combinatory Logic can be expressed by equations or systems of equations. This motivates the study of classes of combinatorial equation systems for which it is possible to provide uniform methods of solution, and the study of classes of systems whose solvability can be proved to be undecidable.

Furthermore, from a different standpoint, a system of equations can be viewed as the specification of a problem in a declarative language, the system's solution being an executable program satisfying the equations, as e.g. in Ref. 1. From such a perspective, since $\lambda$-calculus can be considered as the prototype of any (higher-order) functional programming language, it seems natural to consider the study of combinatorial equations and systems of equations as a theoretical foundation for a kind of synthesis of functional programs and machines.

As an example, we consider the problem of constructing a functional machine $M$ s.t.:

(i) $M$ has a single instruction $F$;

(ii) Any recursive function can be programmed in $M$, i.e. any recursive function can be expressed as an applicative combination of $F$'s;

(iii) The instruction $F$ itself implements a particular given recursive function;

(iv) Some given programs $A_1, ..., A_n$ can be "easily" expressed using $F$. 
Table A. Solvability of equations and systems of equations.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solvability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3 X , M X = \gamma N X)</td>
<td>UNDECIDABLE for semisensible (T) (see 1.1)</td>
</tr>
<tr>
<td>(3 X , M X = \gamma (y \in M):)</td>
<td>ALWAYS SOLVABLE for semisensible (T) (see 1.2)</td>
</tr>
<tr>
<td>- (M = \lambda x_1, \ldots, x_n N_{\gamma M})</td>
<td>OPEN for extensional (T), NEVER SOLVABLE otherwise</td>
</tr>
<tr>
<td>- (M = \lambda x_1, \ldots, x_n N_{\gamma M})</td>
<td>NEVER SOLVABLE for semisensible (T)</td>
</tr>
<tr>
<td>(3 X , M_{\gamma M} = \gamma y_1 (\gamma \in M), , M_{\gamma M} )</td>
<td>CHARACTERIZED (a) (see 1.3)</td>
</tr>
<tr>
<td>(3 X , M_{\gamma M} = \gamma y_1 (\gamma \in M), , M_{\gamma M} )</td>
<td>CHARACTERIZED (b) (see 1.4)</td>
</tr>
</tbody>
</table>

**Regular Systems:**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solvability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3 X_1, \ldots, X_n , M_{X_1, \ldots, X_n} = \gamma y_1 (\gamma \in M), , M_{X_1, \ldots, X_n} )</td>
<td>OPEN for extensional (T), when (M) has more than (n) initial (\lambda)'s</td>
</tr>
<tr>
<td>(3 X_{1, \ldots, X_n} , M_{X_{1, \ldots, X_n}} = \gamma x_1, \ldots, x_n P_{\gamma M} (\gamma \in M), , M_{X_{1, \ldots, X_n}} )</td>
<td>CHARACTERIZED for a suitable distribution of right hand sides</td>
</tr>
<tr>
<td>(3 X_{1, \ldots, X_n} , M_{X_{1, \ldots, X_n}} = \gamma x_1, \ldots, x_n P_{\gamma M} (\gamma \in M), , M_{X_{1, \ldots, X_n}} )</td>
<td>UNDECIDABLE otherwise, when (M) has at most (n) (\alpha) initial (\lambda)'s</td>
</tr>
<tr>
<td>(3 X_{1, \ldots, X_n} , M_{X_{1, \ldots, X_n}} = \gamma x_1, \ldots, x_n P_{\gamma M} (\gamma \in M), , M_{X_{1, \ldots, X_n}} )</td>
<td>as above</td>
</tr>
</tbody>
</table>

(a) Bohm's theorem (b) in \(\beta\)-calculus, separability (c) in \(\beta\Omega\)-calculus

Reducible to the separability problem.

Such a problem can be formulated by the following system, in the unknown \(F\):

\[
\begin{align*}
F \text{ zero } &= y_0 \\
F \text{ success } z &= y_1 \, F \\
F &= A_1 \\
F &= A_2 \\
\cdots \\
F &\, A_{n+1} = A_n \\
F &\, A_0 = K \\
F &\, K = S \\
\end{align*}
\]

(0.1)

where zero and success are respectively the zero and the successor function of some adequate numeral system (Ref.2, §6.4.1). In effect, we have:
- (i) is obviously satisfied;
- (ii) is enforced by the eqs \(n+2, n+3\);
- (iii) is enforced by the eqs 0, 1;
- (iv) is enforced by the eqs 2, 3, \ldots, \(n+1\).

Table A summarizes the state of the art in the study of combinatory equations. In Section 1 we shall illustrate the contents of that table and we shall prove the undecidability of some classes of equations and systems. We shall then introduce (Section 2) a class of systems, which will be called regular systems, as a natural extension of the decidable classes already appearing in the literature. The significance of the class of regular systems shall be emphasized showing how it is difficult to generalize it without running into undecidable problems (Section 4).

Furthermore, we will show (Section 3) how problems and properties of \(\lambda\)-calculus can be described and solved by means of regular systems; for instance, system (0.1), under certain assumptions over left-hand side members of the equations, turns out to be regular, but we shall also examine more classical properties of \(\lambda\)-calculus, concerning e.g. invertibility of terms, numeral systems and fixed point combinators.

We assume the reader is familiar with the basic notions of \(\lambda\)-calculus not explicitly cited in this paper; for a complete treatment of them, see e.g. Refs. 2 and 5.

We shall adopt the following notation: \(\varepsilon = (\xi \neq \eta)\) denotes convertibility, \(\equiv\) denotes identity of objects; \(F(V)\) denotes the set of free variables of a \(\lambda\)-term \(T\); we define:
- \(\text{HNF} = \{\lambda x_1, \ldots, x_n T_1, \ldots, T_m | (\xi \neq \eta) \wedge (T_1, \ldots, T_m \in \Lambda, n, m \geq 0)\}
- \(\text{SOL} = \{M \in \Lambda | \exists N \in \text{HNF} s.t. M = \eta N.\)\)

If \(T = \lambda x_1, \ldots, x_n T_1, \ldots, T_m \in \text{SOL}\), we define the order, degree, head and the \(\text{Bohm tree}\) of \(T\) to be respectively:
- \(\text{ord}(T) = n\), \(\text{deg}(T) = m\), \(\text{head}(T) = \xi\)
- \(BT(T) = \lambda x_1, \ldots, x_n \xi\)

(a) otherwise, if \(T \not\in \text{SOL}: BT(T) = \Omega\) (head \(T\), ord \(T\) and deg \(T\) are not defined here).

Roughly speaking, we say that a \(\lambda\)-theory \(T\) is semisensible (sms) if it never equations two terms \(A\) and \(B\), where \(A \in \text{SOL} \land B \not\in \text{SOL}\).

A path \(\gamma = \langle a_1, \ldots, a_k \rangle | (k \geq 0)\) starting from the root of \(BT(T)\) is a (possibly empty) finite sequence of positive integers uniquely identifying a (pos,non-proper) subtree of \(BT(T)\); roughly speaking, a path \(\gamma\) is defined if it does not meet any \(\Omega\); we will then write \(\gamma \in BT(T)\) and we will denote by \(T_{\gamma}\) the subtree of \(T\) whose \(\text{Bohm tree}\) is identified by \(\gamma\).

**Systems of combinatory equations: some definitions**

A system of equations in \(\lambda\)-calculus will be considered as a pair \(S = (\Gamma, X)\), where \(\Gamma\) is a finite set of formulas of \(\Lambda\) (the equations) and \(X = \{x_1, \ldots, x_n\}\) is a finite set of variables of \(\Lambda\) (the unknowns).

We will denote by \(\Lambda_{T_r}(\mathcal{R}_T)\) the set of left-hand side (lhs) (right-hand side (rhs)) members of the equations in \(\Gamma\).

A solution for the system \(S\) in the \(\lambda\)-theory \(T\) is a set \(\mathcal{V} = \{V_1, \ldots, V_n\}\) of terms (with \(\text{FV}(\mathcal{V}) \cap \text{FV}(\Lambda_{T_r}(\mathcal{R}_T)) = \emptyset\)) such that the substitution \(\{V_1/x_1, \ldots, V_n/x_n\} \Gamma\) makes the equations of \(S\) theorems in \(T\). For simplicity's sake we will denote by the pair \(\mathcal{E} = (U, \mathcal{V})\) a single combinatory equation.

From now onwards the contents of Table A shall be rephrased according to the previous definitions, e.g. the first line of the table will be expressed asking for the solvability of \(\mathcal{E} = \{M \equiv \text{N x}, (x)\}\), where \(x \not\in \text{M.N.}\). Though this might appear as a mere change of notation, it will allow us to simplify the description of important properties of systems.

1. Decidable Combinatory Equations and Systems

In this section we will analyse equations and systems of equations, whose solvability has been studied in the literature, and we will emphasize the relationships existing among them. Moreover, we will identify some classes of systems whose solvability we will prove (Section 4) to be undecidable and some problems which are still open.
1.1. Combinatory Equations

Being not restricted to consider a single unknown, the most general shape for combinatory equations is:

\[ E = (Mx = N, \{x\}), \quad \text{where } x \in M,N. \]  

(1.1)

Statman and Dezani respectively proved the undecidability of the equations, belonging to that class (see also Ref.10): \( E_1 = (Mx = N, \{x\}) ; \quad E_2 = (Mx = I, \{x\}) \).

In Section 4 we will extend their proofs to the most general case.

These results force the assumption of stronger conditions for rhs members of equations. Indeed, one may start asking for the solvability of the equation

\[ E = (Mx = y, \{x\}), \]  

(1.2)

where \( x \in M \) and \( y \) is a variable not occurring in \( M \). In other words, (1.2) is equivalent to asking whether, given \( M \), for every term \( T \) there exists a term \( T_x \) such that

\( MX = T. \)

(1.2)*

To be more explicit: (1.2) is a special case of (1.2)* with \( T = y \) and if \( X \) is a solution of (1.2) then, for every \( T \), \( X = T = (\lambda y . T) \) is a solution of (1.2)*.

The solvability of (1.2) corresponds to the existence of a left inverse for \( M \), i.e. a term \( M^\delta \) s.t. \( BM^\delta = I \), since if \( X \) is s.t. \( MX = y \), then \( M^\delta = \lambda y . X \) and vice versa, if \( M^\delta \) is s.t. \( BM^\delta = I \), then \( X = M^\delta y \) is s.t. \( MX = y \). Such a problem (together with the existence of a left inverse for \( M \), i.e. a term \( M^\delta \) s.t. \( BM^\delta = I \)) has been characterized for \( \lambda \beta \)-calculus in Ref. 11 (see also Ref. 12). More precisely, the equation (1.2) is \( \beta \)-solvable \( \Leftrightarrow \) for some \( N_1, \ldots , N_m \): \( MX = y = xN_1 \ldots N_m \).

The right/left invertibility problem in \( \lambda \beta \)-calculus has been studied in Ref. 13, where sufficient conditions for its solvability have been given, while full invertibility in \( \lambda \beta \eta \)-calculus has been characterized in Ref. 14.

1.2. Combinatory Equations in Extensional Theories: A Remark

The characterization of the solvability of (1.2) can be easily extended to any non-extensional semisensible (sms) theory, but it is still an open problem in the presence of extensionality in the case where \( Mx \) does not reduce to a \( \lambda \)-free term. For such a reason, while considering systems, we will assume the lhs members of the equations to be \( \lambda \)-free.

With this restriction, we will obtain characterizations which hol in any sms theory. The corresponding unrestricted problems, whenever the shape of the considered systems allows their formulation, do not admit any solution in non-extensional theories, but must be considered still open in the presence of extensionality.

1.3. Systems of Equations

The first equation systems for which a method of solution was given have been studied in Böhm's theorem (Ref.6 for \( t = 2 \), Ref.15 for \( t \geq 2 \)):

\[ S = (\Gamma, \{x\}), \quad \text{with } \Gamma = \{ xM_i = y_i \mid i \in \{1, \ldots , l\} \}, \]  

(1.3)

where the \( M_i \)'s are closed \( \beta \eta \)-normal forms and the \( y_i \)'s are arbitrary variables.

More precisely, it was proved that (1.3) is solvable if, modulo \( \alpha \)-conversion, for \( i, k = 1, \ldots , l \):

\[ M_i \beta \eta M_k \Rightarrow y_i \equiv y_k. \]

It comes out that (1.3) can be solved, whenever possible, substituting a Church s-tuple \((\lambda a_1, \ldots , a_n)\) of terms for the unknown \( x \), with \( s \) suitably large. In Ref. 16 it has been proved that it is sufficient to bound \( s \) to \( \mu + 1 \) where \( \mu \) is the maximum of the orders of the \( M_i \)'s.

It is important to note that it is the hypothesis for the \( M_i \)'s to be closed normal forms that allows stating Böhm's theorem in terms of \( \beta \eta \)-convertibility. This is no more sufficient in the general case, as shown by the following examples:

- \( S = (\Gamma, \{x\}) \), with \( \Gamma = \{ xz = y_1, x1 = y_2 \} \), not solvable while \( z \not\equiv \beta \eta I \);  
- \( S = (\Gamma, \{x\}) \), with \( \Gamma = \{ x(\omega \omega) = y_1, xI = y_2 \} \), not solvable while \( \omega \omega \equiv \beta \eta I \) (\( \omega = \lambda x.x \)).

The following notion will enable us to consider the solvability of the system

\[ S = (\Gamma, \{x\}), \quad \text{with } \Gamma = \{ xM_i = y_i \mid i \in \{1, \ldots , l\} \}, \]  

(1.4)

where the \( M_i \)'s are arbitrary terms not containing the variable \( x \).

1.3.1. \( \mathcal{F} \)-indistinctness

The following definitions are needed to introduce the notion of \( \mathcal{F} \)-indistinctness:

- Let \( T_1, T_2 \in \text{HNF} \) and \( \gamma \) be a path \( \gamma \in \text{BT}(T_i), i = 1, 2 \):
  - equivalence between terms:\( T_1 = T_2 \Leftrightarrow \text{head}(T_1) = \text{head}(T_2) \land \text{deg}(T_1) = \text{deg}(T_2) \land \text{ord}(T_1) = \text{ord}(T_2) \).
  - equivalence between terms:\( T_1 \gamma T_2 \iff (T_1 T_2) \gamma T_2 \).

- Let \( \mathcal{F} \subseteq \Lambda \) be a finite set of terms, for every \( F \in \mathcal{F}, \gamma \in \text{BT}(F); \) we say that \( \gamma \) is useful for \( \mathcal{F} \) iff (\( \forall F \in \mathcal{F}, \forall \gamma \in \text{BT}(F) \)):(\( \exists F_1, F_2 \in \mathcal{F} \text{ s.t. } F_1 \gamma F_2 \) \( \land \) (\( \forall \alpha \gamma \forall M, N \in \mathcal{F} : M \alpha = N \).

Definition 1. (\( \mathcal{F} \)-indistinctness, introduced in Refs. 17, 18, see also Ref. 10)

Let \( \mathcal{F} \subseteq \Lambda \) and \( M, N \in \mathcal{F} \); we define the relation \( \gamma \subseteq \mathcal{F} \times \mathcal{F} \), as follows:

\[ M \beta \eta N \Leftrightarrow \exists \mathcal{P} \subseteq \mathcal{F} \text{ s.t. } (\{M, N\} \subseteq \mathcal{P} \land \lnot (\exists \alpha \text{ useful for } \mathcal{P} \)). \]

It comes out that the solvability of the system (1.4) can be reduced to the so-called separability problem, which was characterized in Ref. 7, where the relation of \( \mathcal{F} \)-indistinctness between terms, which was implicitly given, played a role similar to that of \( \beta \eta \)-convertibility in Böhm's theorem.

More precisely: Let \( \mathcal{M} = \{ M_1, \ldots , M_l \} \) be a set of closed terms; the system

\[ S = (\Gamma, \{x\}), \quad \text{with } \Gamma = \{ xM_i = y_i \mid i \in \{1, \ldots , l\} \}, \]  

(1.4)*

where the \( y_i \)'s are arbitrary variables, is \( \beta \eta \)-solvable iff, for \( i, k = 1, \ldots , l \):

\[ M_i \beta \eta M_k \Rightarrow y_i \equiv y_k. \]  

(1.5)

The problem of solvability of (1.4) can then be easily reduced to that of (1.4)* substituting \( \Omega \) for every free variable in the \( M_i \)'s, thus obtaining a system with the shape of (1.4)*.
2. From Self-Application To Regular Systems

An important (and meanwhile restrictive) feature of systems having the shape (1.3,4) is the absence of self-application, the unknown being required to occur exactly once in every equation.

To fill this gap, we shall consider systems in which the unknowns are allowed to appear any number of times in left-hand side members of the equations.

It comes out that the notion of indistinctness is no more sufficient to characterize the solvability of systems with self-applicative features, as shown by the following example:

- \( S = (\Gamma, \{ x \}) \), with \( \Gamma = \{ x(x x) = y_1, x I = y_2 \} \), which is not solvable while \( x(x I) \equiv x(x x)\).

In order to characterize the solvability of systems in which self-application of unknowns appears, we introduce the notion of left-regularity for a system of combinatory equations.

Definition 2. (effective path)
A path \( \gamma \) is said effective for \( \Gamma \in HNF \) iff \( \gamma = <a_1, \ldots, a_k>(k > 0) \land \deg(\Gamma) \geq a_k \).

In self-applicative systems, in order to discriminate external occurrences of the unknowns from internal ones, we shall isolate the set of proper subterms of lhs’s of the equations whose head is an unknown.

Definition 3. (critical subterms)
Given a system \( S = (\Gamma, X) \), we define the set \( CS(S) \) of critical subterms of \( S \):

\[ CS(S) = \{ M \mid M \in L_\Gamma \land \alpha \in BT(M) \land head(M) \not\in X \} \]

Free variables not belonging to the set of unknowns are not involved in any substitution, hence, without loss of generality, they will be considered as undefined objects.

Definition 4.
Given \( S = (\Gamma, X) \), let \( Y = FV(L_\Gamma) \setminus X \). We define \( S_\Omega = (\Gamma_\Omega, X) \), where \( \Gamma_\Omega \) is obtained from \( \Gamma \) substituting \( \Omega \) for the elements of \( Y \).

We are now able to introduce the notion of left-regularity, which is the central issue in characterizing the solvability of self-applicative systems, requiring that external and internal occurrences of the unknown be discriminable; roughly speaking, a subterm of a lhs of any equation must not "collapse" with a lhs.

Definition 5. (left-regularity)
Let \( S = (\Gamma, X) \):

- We first define a new equivalence relation \( \equiv_S \subseteq (L_\Gamma \cup CS(S)) \times (L_\Gamma \cup CS(S)) \):

\[ U \equiv_S V \Leftrightarrow head(U) = head(V) \land \exists P \neq \emptyset \subseteq L_\Gamma, P \subseteq CS(S) \text{ s.t.:} \]

\[ (U, V) \subseteq P_1 \cup P_2 \land \exists x \text{ useful for } P_1 \cup P_2 \text{ and effective for } P_1 \] \( . \)

- We say that \( S \) is left-regular iff the following conditions are both satisfied:

(i) \( \exists L \in L_\Gamma, N \in CS(S) \text{ s.t. } L \equiv_S N \); (ii) \( V U, V \in L_\Gamma, U \equiv_S V \Rightarrow deg(U) = deg(V) \).

Example 1.
- The system \( S = (\Gamma, \{ x \}) \) with \( \Gamma = \{ x(x \Omega) = y_1, x x = K \} \) is not left-regular since \( x \Omega \equiv_S x x \).
- The system \( S = (\{ x(x K) = x, x x = x B \}, \{ x \}) \) is left-regular.

We can "throw away unnecessary information" from the system \( S \) considering one of its approximations \( S' \). If \( S' \) is solvable then also \( S \) is solvable.

Definition 6.
- Let \( U, V \in \Lambda \); we say that \( U \) approximates \( V \) (we write \( U \subseteq V \)) if \( BT(U) \subseteq BT(V) \).
- Let \( M \in \Lambda \); we define \( \text{Approx}(M) = \{ N \in \Lambda \mid N \subseteq M \} \).
- \( \{ \text{Approx}(S) = \{ S' \} \subseteq (\Gamma, X) \land \Gamma = \{ L = R \mid L = R \in \Gamma \land M' \in \text{Approx}(M) \} \} \).

The following theorem characterizes the solvability of a class of systems exhibiting self-application of unknowns; the solvability of such systems has been studied in Refs. 17, 18, 19 and then characterized in Ref. 20 in the special case where rhs are pairwise distinct – the X-separability problem – and in Ref. 10 in the general case.

Theorem 1. Let \( T \) be an sms theory and \( S = (\Gamma, X) \) a system such that:

(i) Any equation of \( \Gamma \) has the shape \( x_1 \ldots x_m = y \), where \( x \in X \) and \( y \in (X \cup FV(L_\Gamma)) \);
(ii) For every pair of equations \( L_1 = R_1, L_2 = R_2 \) in \( S_\Omega \):

\[ head(R_1) = head(R_2) \Rightarrow L_1 \text{ and } L_2 \text{ are } L_\Gamma \text{-indistinct} \]

Then \( S \) is solvable in the theory \( T \) iff there exists \( S' = (\Gamma', X) \in \text{Approx}(S_\Omega) \) such that:

- \( S' \) is left-regular;
- for every pair of equations \( L_1 = R_1, L_2 = R_2 \) in \( S' \):

\[ L_1 \text{ and } L_2 \text{ are } L_\Gamma \text{-indistinct } \Rightarrow head(R_1) = head(R_2) \]

Proof.

The most important issues in proving the theorem shall be discussed in the next subsection. For a more detailed proof see Ref. 10.

2.1. From the Proof of Theorem 1 to Regular Systems: Some Remarks

2.1.1. Building up the solution for a self-applicative system

In order to characterize in a constructive way the solvability of a self-applicative system \( S \) we define a method which allows us to keep control of the consequences of self-application, since multiple occurrences of the unknowns, appearing at different depths in the ββ trees of lhs members of the equations, must be substituted by the same terms. Indeed, the left-regularity condition enables us to take into account the effects of the cited substitution over all the subterms of lhs members of the equations which are involved in it.

To be more pragmatic, the central issue in defining a method for solving equation systems lies in the shape of the the term constituting the solution to the problem itself. In our approach, the solution of a self-applicative system will be constructed in the following way: given
2.2. Regular Systems

Definition 7.
Let \( S = (\Gamma, X) \) a system such that \( S_\Omega \) is left-regular, \( \Gamma = \Gamma_1 \cup \Gamma_2 \) and \( X = \{ x_1, \ldots, x_n \} \). \( S \) is said to be regular if

(i) Any equation of \( \Gamma_1 \) has one of the following shapes
   - \( x N_1 \ldots N_m = y N_1 \ldots N_m \)
   - \( x N_1 \ldots N_m = y x_1 \ldots x_n z_1 \ldots z_q \)
   where \( x, y \in X \) and \( x_1 \ldots x_n z_1 \ldots z_q \) is an effective path for \( x N_1 \ldots N_m \) s.t. the variables appearing along it do not belong to \( FV(x N_1 \ldots N_m) \).

(ii) For every pair of equations \( L_1 = R_1 \in \Gamma_1 \), \( L_2 = R_2 \in \Gamma_2 \) head(\( R_1 \)) \( \neq \) head(\( R_2 \)).

(iii) Any equation of \( \Gamma_2 \) has the shape
   - \( x N_1 \ldots N_m = N \)
   where \( x \in X \) and \( N \) is a \( \beta_\eta \) normal form.

(iv) For every pair of equations \( L_1 = R_1 \in \Gamma_1 \), \( L_2 = R_2 \in \Gamma_2 \):
   - \( L_1 \) and \( L_2 \) are \( \mathcal{L}_{\Gamma_2^\prime} \)-indistinct \( \Rightarrow \) deg(\( R_1 \)) \( \neq \) deg(\( R_2 \)) - ord(\( R_2 \)).

(v) For every pair of equations \( L_1 = R_1 \in \Gamma_1 \), \( L_2 = R_2 \in \Gamma_2 \), \( R_1 \) and \( R_2 \) are \( \mathcal{R}_{\Gamma_2^\prime} \)-indistinct \( \Rightarrow \) \( L_1 \) and \( L_2 \) are \( \mathcal{L}_{\Gamma_2^\prime} \)-indistinct.

Example 2.
The following system is regular
\[
S = (\{ f(xab) = y_0, \\
    f(x,bb) = y_1 f z, \\
    ff = K, \\
    fK = S, \\
    fS = y_2 \}, \{ f \})
\]

On the other hand, the following systems are not regular, since in \( S_1 \) the first equation does not satisfy condition (iii) of Definition 7, while \( S_2 \) is not left-regular since it does not satisfy condition (ii) of Definition 5:

- \( S_1 = (\{ f f = f B, f K = 1, \{ f \} \}) \)
- \( S_2 = (\{ f g f = B, f g = 1, \{ f \} \}) \)

Theorem 2. (Main theorem)
Let \( T \) be an msms theory and \( S = (\Gamma, X) \) a regular system.
Then \( S \) is solvable in the theory \( T \) iff
for every pair of equations \( L_1 = R_1 \in \Gamma_1 \), \( L_2 = R_2 \in \Gamma_2 \):
\( L_1 \) and \( L_2 \) are \( \mathcal{L}_{\Gamma_2^\prime} \)-indistinct \( \Rightarrow \) \( R_1 \) and \( R_2 \) are \( \mathcal{R}_{\Gamma_2^\prime} \)-indistinct.

Sketch of the proof:
(\( \Rightarrow \)): From Theorem 1.
(\( \Leftarrow \)): As in Ref. 17, given the fresh variables \( u_1, \ldots, u_n \), substitute for the variable \( x_i \) the term \( u_i u_1 u_2 \ldots u_n \) thus obtaining a new system in the unknowns \( u_1, \ldots, u_n \). Then, use theor.1, Böhm-out technique and make the suitable substitutions.
Note that for regular systems the conditions for solvability with fresh variables in rhs members of the equations or with combinators are the same. This is not the case for non-regular systems.

Moreover, by means of equations having the shape \(x N_1...N_m = y x_1...x_n z_1...z_r\), we are able to Böhm-out the variables \(z_1,...,z_r\) as well as any term we substitute for them in \(x N_1...N_m\).

By means of the notion of regularity it is possible to give partial characterizations for the invertibility problem in extensional theories (see also Ref. 13).

**Theorem 3.** (left-invertibility)

Let \(T\) be an extensional smx theory and \(M = \lambda y x_1...x_n y M_1...M_m\), where, for \(i = 1,...,t, M_i\) is \(\lambda\)-free and \(\text{head}(M_i) = x_i\). Consider the system \(S = \{M_i = y_i, i \in \{1,...,t\}, x\}\) where the \(y_i\)'s are pairwise distinct. Then:

- \(M \text{ is } T\)-left-invertible iff there exists a left-regular approximation of \(S\), whose \(x\)'s are distinct.

**Proof.** From Ref. 10, §2.7, Ref. 17, §9.1 and Theorem 2.

**Theorem 4.** (right-invertibility)

Let \(T\) be an extensional smx theory and let \(M = \lambda x z_1...z_r x M_1...M_m\) be s.t. the \(z_i\)'s occur only as leaves in \(BT(x M_1...M_m)\) and, denoting \(E = (x M_1...M_m = y, \{x\})\), the equation \(E\) is left-regular. Then:

- \(M \text{ is } T\)-right-invertible iff for \(i = 1,...,r, z_i \in \text{FV}(BT(E/\text{FV}(x M_1...M_m) - \{x,z_1,...,z_r\}) (x M_1...M_m)))\).

**Proof.** (\(\Rightarrow\)): trivial; (\(\Leftarrow\)): From Theorem 2.

### 3. Solving Problems By Means Of Regular Systems

We will now exhibit some examples about problems which can be solved by means of regular systems.

#### 3.1. A Numerical System

Let \(O = \lambda x y . y, W = \lambda x y . x y y\). As known, the sequence \(O, W, W(WO),...\) (Shap numerals\(^{23}\)) is an adequate numeral system. Our aim is to give a proof for this, providing a term \(I_n\) (see Ref. 22) which gives a correspondence between the cited sequence and Church numerals, i.e. such that, being \(n\) the \(n\)-th element of the sequence, \(I_n = n\), the \(n\)-th Church numeral.

Since \(O = \lambda x . x, W = \lambda x u . u U\), the problem is solved if we find \(H\) which, whenever applied to itself a number of times and successively applied to \(U\), gives the Church numeral corresponding to the number of times it has been applied to itself.

This amounts to solving the system of equations, in the unknowns \(h, u:\)

\[
\begin{align*}
  h x (h y) &= h (x \circ c x) \\
  h x u &= x
\end{align*}
\]

where \(c\) is Church's successor combinator. In fact, if \(H^{\prime}\) solves (a), then \(H = H^\prime O\) is a solution for our problem. We consider the system:

\[
\begin{align*}
  h x (h y) &= y, h x \\
  h x u &= y, h u x (h y) \\
  h x u &= y, h u \text{xuu}
\end{align*}
\]

and we note that from (b) we obtain (a) substituting \(\lambda uv. u(v, v)\) for \(y\) and \(\lambda uv. v\) for \(y\).

The solvability of (b) is a trivial consequence of Theorem 2, since by it we are able to solve e.g. (c). A possible solution is found taking \(h = z z\) and considering the new system, in the unknowns \(z, u:\)

\[
\begin{align*}
  z z x (z z y) &= y, (z z) x \\
  z z u &= y, (z z) x
\end{align*}
\]

and substituting \(\lambda \text{rst.}(\lambda \text{abcd} y, y (bb))\text{rst for z and } \lambda \text{abcd} y, c\) for \(u\).

Note that, starting from the slightly different system:

\[
\begin{align*}
  h x (h y) &= h (x \circ c y) \\
  h x u &= x
\end{align*}
\]

we can also obtain, following a similar construction, a term \(H\) which, composed with itself a number of times, preserves the memory of the number of times it has been composed with itself.

#### 3.2. Make Your Own Fixpoint Combinator

We recall (Ref.2, §6.1) that a fixed point combinator is a term \(M\) s.t. \(\forall F, M F = F (MF)\).

Two examples of fixed point combinators are Turing's one, \(Y_T = (\lambda x . f (x f) x f) (\lambda x . f (x f) x f)\), and Curry's one, \(Y_C = \lambda x . (\lambda x . x f) (\lambda x . x f)\).

Obviously, they are both solutions of the equation, in the unknown \(y\):

\[
y x = y (x y)\,.
\]

but, in order to recover constructivity, we may ask whether \(Y_C\) and \(Y_T\) can be obtained, via the same method of solution, from different equations equivalent to (3.1): in effect we have (see also Refs. 3 and 4 for different methods for constructing Turing's and Curry's fixpoint combinators):

(i) substituting \(z z\) for \(y\) in (3.1) we obtain the equation in the unknown \(z\):

\[
z z x = x (z z x)
\]

(ii) substituting \(x z (x z x)\) for \(y\) in (3.1) we obtain the equation in the unknown \(z\):

\[
x z (x z x) = x (x z (x z x))
\]

whose solution is found substituting \(\lambda x . x f) (x f) x f\) for \(z\), thus obtaining \(Y_T\).

Indeed, it is easy to verify that the following equations in the unknown \(z\) are regular:

- \(z z x = y z x\) from which we obtain (3.2) substituting \(\lambda x . x f) (x f) x f\) for \(y\)
- \(x z (x z x) = y (x z x)\) from which we obtain (3.3) substituting \(\lambda x . x f) (x f) x f\) for \(y\).
In order to find Turing's and Curry's fixed point combinators, we can solve the equations substituting y for the unknown z and then make the mentioned substitution for y. More generally, given the fixed point equation (3.1), any way of rephrasing it as a regular equation leads us to construct a fixed point combinator. As an example, substitute for y in (3.1) the term zM1...Mw, where M1,...,Mw are arbitrary terms; solving the equation z zM1...Mw = y zM1...Mw x and successively taking y = λab1...b2.x.c.c(c(a ab1...b2 e)) we succeed in generalizing the theorem in (Ref.2, §6.5.4).

3.3 Recursive Schemes and Functional Programming

Let us turn back to considering the solvability problem for the system (0.1).

By Theorem 2, if we assume that in (0.1)

- succes is such that it is possible to Böhm-out z from succes z;
- The set {A1,...,An, K, zero, succes, Ω} is distinct,

then the considered system is β - solvable and its solution is the program we were looking for.

As an example, we take Berarducci's mmalb24, where zero = λab b and succes = λab.abb, and we exhibit the construction of the solution for the system in Example 2, as described in §2.1.1:

\[ S = \{ f(\lambda ab. b) = y, \]
\[ f(\lambda b. bzb) = y_1 f z, \]
\[ fK = S, \]
\[ fS = y_2, (f f) \}

(3.4)


- Step 1: substitute for f the term
  G G where G = λab. bu.u2.u4.u5.u6.ab,
  thus obtaining a new system:
  \[ S_0 = \{ u_1 z u_2 u_3 u_4 u_5 u_6 G (\lambda b. bzb) = y_1 (GG) z, \]
  \[ u_1 u_2 u_3 u_4 u_5 u_6 G (\lambda b. bzb) = y_1 K, \]
  \[ u_1 u_2 u_3 u_4 u_5 u_6 G S = y_2, \}

- Step 2: substitute for u1 the term U1 = λab cd. dabc,
  thus obtaining a new system:
  \[ S_0 = \{ u_1 z u_2 u_3 u_4 G(\lambda ab. b) = y_0, \]
  \[ u_1 z u_2 u_3 u_4 u_5 u_6 G (\lambda b. bzb) = y_1 (GG) z, \]
  \[ u_1 U_1 u_2 u_3 u_4 u_5 u_6 G U_1 u_2 u_3 u_4 u_5 u_6 (GG) = K, \]
  \[ u_1 U_1 u_2 u_3 u_4 u_5 u_6 G K = S, \]
  \[ u_1 U_1 u_2 u_3 u_4 u_5 u_6 G S = y_2, \}


The head variables of left-hand side members of the equations are now mutually different, so we can make the suitable substitutions in order to reconstruct right-hand sides:

\[ U_2 = \lambda a_1...a_4 y_0 \text{ for } u_2 \]
\[ U_3 = \lambda a_1...a_4 y_1(\lambda b_2) b_1 \text{ for } u_3 \]

- Hence a possible solution for the system (3.4) is
  \[ F = \lambda b. b u_1 u_2 U_3 U_4 U_6 (\lambda cd. d U_2 U_3 U_4 U_5 U_6 cd) b. \]

4. Undecidability Results

In this section we shall consider some classes of combinatory equations and systems for which we shall prove the solvability problem to be undecidable. These results confirm that meaningful generalizations of the class of regular systems turn out to be not decidable.

Note also that in the proofs of both the following theorems, we shall make use of non-regular systems.

Eliminating the condition of regularity, the solvability of an equation between normal forms is not decidable:

**Theorem 5.**

Given the β(βη)-normal form N, the solvability of the equation, in the unknown x,

\[ M x = N, \]

where M is a β(βη)-normal form and x ∈ M, is not decidable in λβ(λη)-calculus.

**Proof.** We shall reduce the solvability of (4.1) to the halting problem for recursive functions.

Since N is a normal form, then for suitable u,v:

\[ N = λt_1...t_n t_1...T_n, \]

where \( t \in \{ t_1,...,t_n \} \cup \text{FV}(N); \)

As in Ref. 25, for each Gödel number e we construct, uniformly in e (Ref. 2, §8.2~4), a term P_e, s.t., with the usual notation of recursion theory:

\[ \varphi_e(e) \text{ converges } \Rightarrow P_e = I \land \varphi_e(e) \text{ diverges } \Rightarrow P_e \notin \text{ SOL} \]

and we take the term P_{e_1} obtained from P_e replacing each redex (λx.Y)X by t (λx.Y)X where t is a fresh variable not occurring in N.

**Proof for λβ-calculus:** take a leaf in BT(N), labelled with \( \lambda z_1...z_n (n \geq 0), \) and let N' be obtained from N replacing that label with \( \lambda z_1...z_n \) P_{e_1}(t ζ). Let then M = λt.N'.

It is easy to verify that:

\[ M x = N \text{ is } \beta - \text{solvability } \Rightarrow \varphi_e(e) \text{ converges. } \]

**Proof for λβη-calculi:** let a,b be variables not occurring in N and take

\[ M = \lambda t_1...t_n a b t_1...T_n(t a) (P_{e_1} b); \text{ then } \]

\[ M x = N \text{ is } \betaη - \text{solvability } \Rightarrow \varphi_e(e) \text{ converges. } \]

The following theorem proves that dropping condition (v) from the definition of regular system leads to an undecidable problem (see also §2.1.3).

**Theorem 6.**

The solvability of a system \( S = (Γ, X) \) such that any equation of Γ has the shape

\[ x_1 N_m = y \text{ where } x, y \in X \text{ and } y \in (X \cup \text{FV}(L_Γ)) \text{ is not decidable in } \lambda β - \text{calculus.}\]

**Sketch of the proof.** Let \( P_e \) and \( P_{e_1} \) be as in the proof of Theorem 5 and let

\[ S = (Γ, \{ t; x \}), \text{ with } Γ = \{ t P_{e_1}; z = y, t (P_{e_1}) = y \}, \]

which contradicts condition (v) of Def.2;
then \( S \) is \( \beta \)-solvable \( \iff \varphi_s(e) \) converges.

\((\Leftarrow)\): If \( \varphi_s(e) \) converges, then the substitution \([I/1, y/z]\) is a solution for \( S \).

\((\Rightarrow)\): Let \([A/x, B/z]\) be a solution for \( S \). We will have: \( A = \lambda x_1 \ldots x_n \cdot \xi y_1 \ldots y_k \) \( (a, b \geq 0) \); it is easy to verify that we must have \( a = 1, \xi = x_1, \) hence: \( A = \lambda x \cdot x y_1 \ldots y_k \ (b \geq 0) \); substituting into the second equation and taking into account the first one we must have:

\[ A = \lambda y \cdot y y_1 \ldots y_k = \]

which clearly holds iff \( b = 0 \). It follows that \( A = I \), hence \( P_e \in \text{SOL} \) and \( \varphi_s(e) \) converges.\( \Box \)

5. Concluding Remarks

Summarizing, we introduced the class of regular systems as a decidable class of systems of combinatory equations which seems to generalize in a significant way the ones already appearing in the literature. Regular systems have enough expressive power to be used as the core of an equational programming language, in which the compiler is the algorithm constructing the system’s solution. Moreover, such class of systems seems not to easily generalizable in meaningful ways without running into undecidable solvability problems.

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