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A Characterization of Weakly Church–Rosser Abstract Reduction Systems That Are Not Church–Rosser[1]

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Basic properties of rewriting systems can be stated in the framework of abstract reduction systems (ARS). Properties like confluence (or Church–Rosser, CR) and weak confluence (or weak Church–Rosser, WCR) and their relationships can be studied in this setting: as a matter of fact, well-known counterexamples to the implication WCR $\Rightarrow$ CR have been formulated as ARS. In this paper, starting from the observation that such counterexamples are structurally similar, we set out a graph-theoretic characterization of WCR ARS that is not CR in terms of a suitable class of reduction graphs, such that in every WCR not CR ARS, we can embed at least one element of this class. Moreover, we give a tighter characterization for a restricted class of ARS enjoying a suitable regularity condition. Finally, as a consequence of our approach, we prove some interesting results about ARS using the mathematical tools developed. In particular, we prove an extension of the Newman's lemma and we find out conditions that, once assumed together with WCR property, ensure the unique normal form property. The Appendix treats two interesting examples, both generated by graph-rewriting rules, with specific combinatorial properties. © 2001 Elsevier Science

1. INTRODUCTION

Abstract reduction systems (ARS) are abstract structures for studying general properties of rewriting systems. An ARS is just a set equipped with a set of binary relations. Despite this generality many aspects of rewriting systems can be approached in the theory of ARS, such as confluence and normalization. Moreover, important results have been stated and proved in this setting (examples are Newman's lemma and the Hindley–Rosen theorem [1]). Also the relationships between the Church–Rosser (CR or confluence) and weak Church–Rosser (WCR) properties can be studied in the ARS setting. Indeed, well-known counterexamples of WCR $\Rightarrow$ CR were abstractly described as ARS by Rosser, Newman, Hindley, Klop, and others [2, 5, 9, 7] (Fig. 1 and 2, displayed below, show some of these counterexamples).

In this work, we study the structure of WCR ARS that are not CR, looking for a graph-theoretic characterization of such ARS in terms of a suitable class of reduction graphs, such that in every WCR not CR ARS, we can embed at least one element of such a class.

The main result of this work is that there exists, indeed, a family $\mathcal{BS}$ of WCR not CR reduction graphs such that, for every WCR not CR ARS $\mathcal{A}$, there is an element of $\mathcal{BS}$ that can be "embedded" in $\mathcal{A}$ for a suitable notion of embedding. This notion of embedding is reminiscent of the well-known notion of homeomorphic subgraph (*minor*) in graph theory (see for example [10]). The family $\mathcal{BS}$ strongly differs, in its general structure, from the above-mentioned family of well-known counterexamples. It is therefore natural to ask what additional conditions can enforce a better-defined shape to WCR not CR ARS. We answer this question by singling out some natural restrictions that, once assumed, strongly specify the structure of a WCR not CR ARS.

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The rest of the paper is organized as follows. In Section 2, we will introduce some basic terminology about ARS. In particular, we define the central properties analyzed in this work, namely confluence and weak confluence. Moreover, inspired by the notion of graph homomorphism, we introduce a notion of embedding for ARS. An embedding is an injective map that, roughly speaking, preserves *compatibility* (i.e., to have a common reduct) and multistep reductions. In Section 3, we will introduce the notion of *sink*, which turns out to be an useful tool in the analysis of WCR not CR ARS. Then we will prove the main technical result, the Bolt Lemma, which gives a general characterization of WCR not CR ARS. This result will be frequently used in proofs in the following sections. In Section 4, we will define a hierarchy of ARS: this hierarchy is intuitively related to *how CR* a WCR ARS is. We give a stronger characterization for more restricted families of ARS. In particular, we will show that in finite WCR not CR ARS we can always embed the Hindley ARS (Fig. 1) and for a natural family of ARS, strongly finite dimensional ARS (SFD), we can strongly specify their structure by means of a finite class of reduction graphs, $\mathcal{F}$, such that for every SFD ARS we can embed in it an element of $\mathcal{F}$. $\mathcal{F}$ essentially corresponds to classical examples of WCR $\not\Rightarrow$ CR. Finally, in Section 5, applying the mathematical tools developed, in particular the Bolt Lemma, we prove some interesting properties about ARS. In particular:

1. an extension of the Newman's lemma (and hence a new proof for it) that holds also for infinite ARS;
2. we find out conditions that, once assumed together with the WCR property, imply the unique normal form property; and
3. an estimate of the number of normal forms in finite WCR ARS.

In the Appendix, we present a detailed analysis of two reduction graphs introduced in this paper. These examples show that, on some specific structures, the topic of this work gives rise also to very concrete combinatorial problems.

2. ABSTRACT REDUCTION SYSTEMS

In this section we recall some basic definitions about ARS. An extensive treatment of ARS can be found in [8].

DEFINITION 2.1. An *abstract reduction system* (ARS) is a pair $\mathcal{A} = \langle A, \{\rightarrow_\alpha\}_{\alpha\in\mathcal{I}}\rangle$ where A is a set and $\{\rightarrow_\alpha\}_{\alpha\in\mathcal{I}}$ is a set of binary relations on A, also called *reduction* or *rewriting relations*. In this work, we consider ARS with only one rewrite relation and write $\langle A, \rightarrow\rangle$.

DEFINITION 2.2. To indicate that $(a, b) \in\rightarrow$, we use the infix notation $a \rightarrow b$. $a \rightarrow b$ is called a *reduction step* (elementary step) and b is called a *one-step reduct* of a. The *inverse relation* of $\rightarrow$ is written $\rightarrow^{-1}$ or $\leftarrow$.

As usual, transitive and reflexive closure is denoted by $\stackrel{*}{\rightarrow}$. If a $a \stackrel{*}{\rightarrow} b$, we call b a *reduct* of a. The equivalence relation generated by $\rightarrow$, also called the *convertibility relation*, is denoted by $=$.

The following definition introduces a notion that will be extensively used in the sequel:

DEFINITION 2.3. Let $\mathcal{A} = \langle A, \rightarrow\rangle$ be an ARS. Two elements $a, b \in A$ are *compatible* or *joinable*, notation $a \downarrow b$, if they have a common reduct.

DEFINITION 2.4. Let $\mathcal{A} = \langle A, \rightarrow\rangle$ be an ARS.

1. We say that $a \in A$ *is a normal form* if there is no b such that $a \rightarrow b$. Further $a \in A$ *has a normal form* if $a \stackrel{*}{\rightarrow} b$ for some normal form $b \in A$.
2. $\mathcal{A}$ (or $\rightarrow$) is *weakly normalizing* (WN) if every $a \in A$ has a normal form.
3. $\mathcal{A}$ (or $\rightarrow$) is *strongly normalizing* (SN) if every reduction sequence is finite. Sometimes this property is indicated saying that $\rightarrow$ is *terminating* or *noetherian*.
4. We say that $\mathcal{A}$ (or $\rightarrow$) has the *unique normal form property* (UN) if $\forall a, b \in A.(a = b)$ and a, b in normal form $\Rightarrow a \equiv b$.
5. We say that $\mathcal{A}$ (or $\rightarrow$) is SN^{-1} if the inverse relation $\leftarrow$ is SN.

6. $\mathcal{A}$ (or $\to$) is *finitely branching* or *locally finite* (FB) if $\forall a \in A$ the set of one-step reducts of a, $\{b \in A \mid a \to b\}$, is finite. If the inverse reduction $\leftarrow$ is FB, we say that $\mathcal{A}$ (or $\to$) is FB^{-1}.

7. $\mathcal{A}$ (or $\to$) is *increasing* (Inc) if there is map $|\ |: A \to \mathbf{N}$ such that $\forall a, b \in A. a \to b \Rightarrow |a| < |b|$.

8. $\mathcal{A}$ (or $\to$) is *inductive* (Ind) if for every reduction sequence $a_0 \to a_1 \to \cdots$ there is an $a \in A$ such that $a_n \stackrel{*}{\to} a\ \forall n \in \mathbf{N}$.

DEFINITION 2.5. Let $\mathcal{A} = \langle A, \to_A \rangle$ and $\mathcal{B} = \langle B, \to_B \rangle$ be two ARS. Then $\mathcal{A}$ is a *sub-ARS* of $\mathcal{B}$, notation $\mathcal{A} \subseteq \mathcal{B}$, if:

1. $A \subseteq B$;
2. $\to_A$ is the restriction of $\to_B$ on A, i.e., $\forall a, b \in A.(a \to_A b) \Leftrightarrow (a \to_B b)$; and
3. $\mathcal{A}$ is closed under $\to_B$, i.e., $\forall a \in A.(a \to_B b \Rightarrow b \in A)$.

The ARS $\mathcal{B}$ is called an *extension* of $\mathcal{A}$.

DEFINITION 2.6. Let $\mathcal{A} = \langle A, \to \rangle$ be an ARS and $a \in A$. The *reduction graph of a*, notation $\mathcal{G}(a)$, is the smallest sub-ARS of $\mathcal{A}$ containing a. We also define inductively the *set of reducts of a* as follows:

1. $\Delta^0(a) = a$.
2. $\Delta^{n+1}(a) = \{b \in A \mid \exists c \in \Delta^n(a).c \to b\}$.
3. Finally we define the set of all reducts of a, $\Delta^*(a) = \bigcup_{n \in N} \Delta^n(a)$.

We usually write $\Delta(a)$ for $\Delta^1(a)$.

It is easy to see that $\mathcal{G}(a) = \langle \Delta^*(a), \to \rangle$. By abuse of notation, we usually write $\mathcal{G}(a)$ to indicate either the reduction graph of a or the set of reducts of a.

DEFINITION 2.7. The reduction relation $\to$ is called:

1. *weakly confluent* or *weakly Church–Rosser* (WCR) if $\forall a, b, c \in A. a \to b \wedge a \to c \Rightarrow b \downarrow c$.
2. *confluent* or *Church–Rosser* (CR) if $\forall a, b, c \in A. a \stackrel{*}{\to} b \wedge a \stackrel{*}{\to} c \Rightarrow b \downarrow c$.

Remark 2.1. By definition, CR $\Rightarrow$ WCR. We show in Figs. 1 and 2 some well-known examples of abstract reduction systems that show that the reverse implication does not hold. We call $\mathcal{F} = \{\mathcal{H}, \mathcal{K}, \mathcal{N}, \mathcal{O}\}$ the set of such reduction graphs. Note that $\mathcal{O}$, in some sense, blends the structures of $\mathcal{K}$ and $\mathcal{N}$.

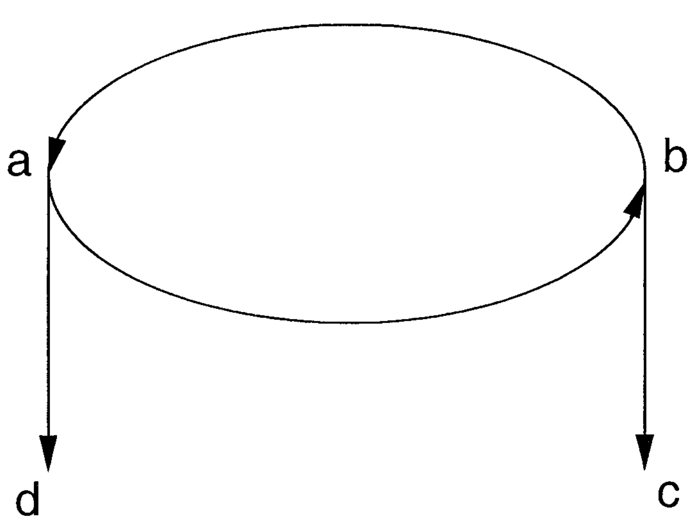

FIG. 1. $\mathcal{H}$, the best known example of ARS WCR not CR, given by Hindley [5].

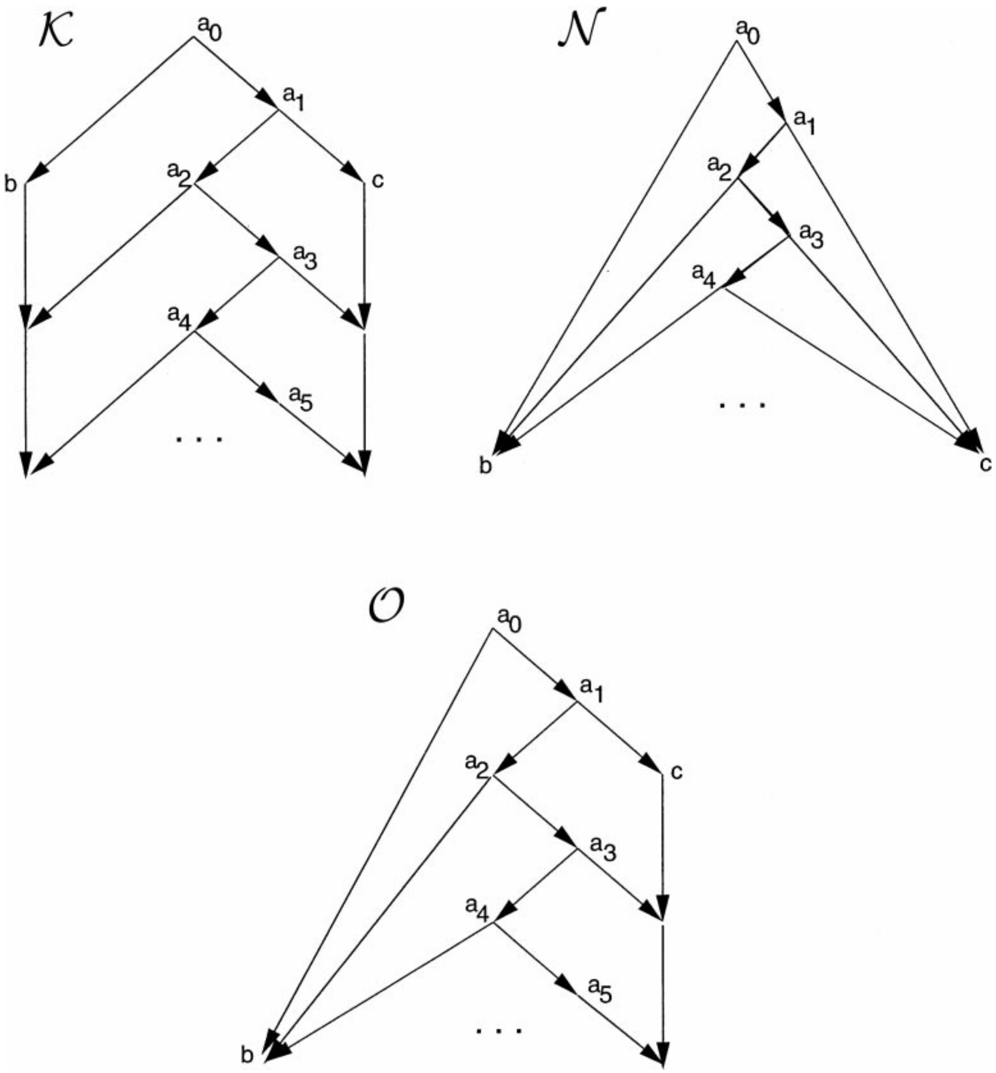

FIG. 2. Three counterexamples that show that WCR does not imply CR, essentially due to Klop [7] and Newman [9].

NOTATION 2.1. In order to improve readability, we call all ARS that are WCR but not CR *strictly WCR* (SWCR).

Many other examples of SWCR ARS have essentially the same structure as one of the ARS in $\mathcal{F}$, except for apparently irrelevant details. Therefore, it is natural to ask whether the reduction graphs in $\mathcal{F}$ are, in some sense, the only possible examples of SWCR ARS. To avoid trivial negative answers, we give a treatment of this question in a way similar to the "forbidden minors" approach in graph theory [10]. To do this, we introduce the following notion of *embedding* of an ARS $\mathcal{A}$ into another ARS $\mathcal{B}$. Observe that we require in particular that the embedding preserves compatibility and incompatibility.

DEFINITION 2.8. Let $\mathcal{A} = \langle A, \to \rangle$ and $\mathcal{B} = \langle B, \to \rangle$ be two ARS. A map $h\colon A \to B$ is an *embedding* if the following conditions hold:

1. h is injective;
2. $\forall a, a' \in A. a \stackrel{*}{\to} a' \Leftrightarrow h(a) \stackrel{*}{\to} h(a')$; and
3. $\forall a, a' \in A. a \downarrow a' \Leftrightarrow h(a) \downarrow h(a')$.

If there exists an embedding $h\colon A \to B$, we say that $\mathcal{A}$ is *embeddable* in $\mathcal{B}$.

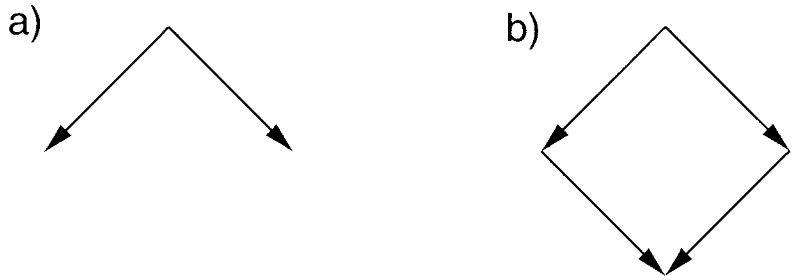

FIG. 3. The ARS (a) is not embeddable in the ARS (b).

We observe that, by 2 and 3, if $\mathcal{A}$ is embeddable in $\mathcal{B}$ and $\mathcal{A}$ is not CR, so is $\mathcal{B}$ (see Fig. 3).

EXAMPLE 2.1. It is easy to see that one can embed the ARS $\mathcal{N}$ (Fig. 2) in the ARS $\mathcal{K}'$ (Fig. 4), simply following the labeling in the pictures.

We can now reformulate our main question as follows: given a SWCR ARS, is it always possible to embed in it one of the reduction graphs in $\mathcal{F}$? Formulated like this, the answer is negative and we will show appropriate counterexamples. However, we will identify in the next section a wider class $\mathcal{BS}$ of reduction graphs that have the property that for each SWCR ARS, there is an element of $\mathcal{BS}$ that is embeddable in it.

3. BOLTS AND SINKS

In this section, we give some technical definitions and prove the main technical result.

DEFINITION 3.1. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS and $a \in A$. We define the set of *compatible* (notation *Comp*(a)) and *strongly compatible,* or *sink* (notation *Sink*(a)), elements with a as follows:

- $Comp(a) = \{x \in A \mid x \downarrow a\}$, and
- $Sink(a) = \{x \in A \mid x \downarrow a \,\&\, (\forall y \in A.x \downarrow y \Rightarrow y \downarrow a)\}$.

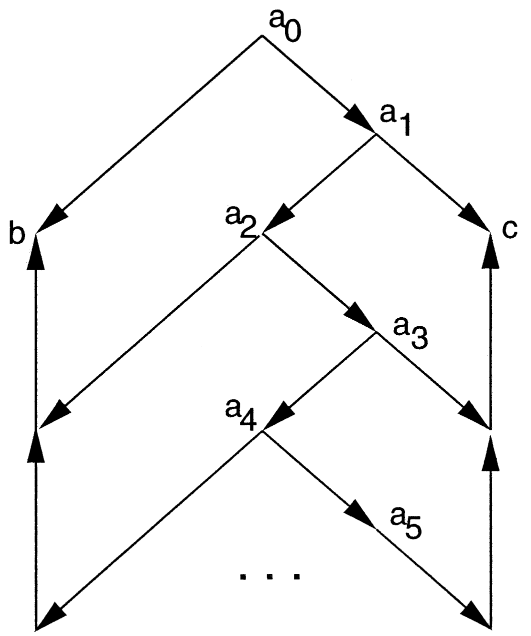

FIG. 4. The ARS $\mathcal{K}'$ [7] in which $\mathcal{N}$ is embeddable.

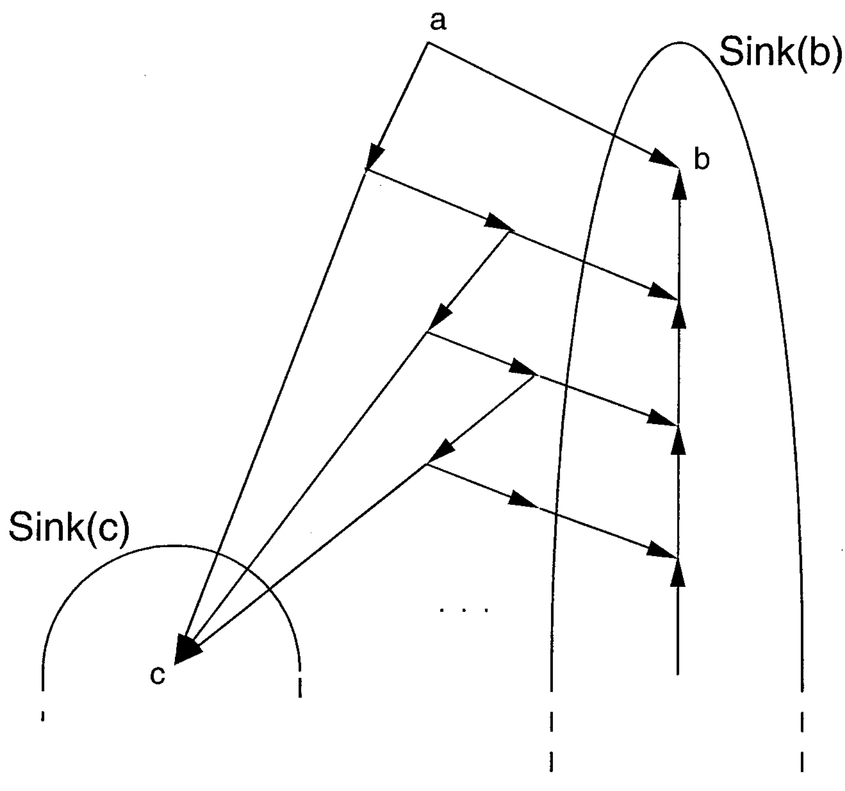

FIG. 5. An example of sinks.

It is easy to check that $Sink(a) = \{x \in Comp(a) \mid Comp(x) \subseteq Comp(a)\}$, which gives a more intuitive characterization of sinks. Moreover, we observe that for every a, $Sink(a) \neq \emptyset$, because we always have that $a \in Sink(a)$. Sometimes we use the notation $b \to Sink(a)$ instead of $\exists a' \in Sink(a).b \to a'$.

EXAMPLE 3.1. Figure 5 shows a reduction graph in which we highlight two sinks.

EXAMPLE 3.2. Figure 6 shows an instructive example of a SWCR ARS with two sinks displayed. This ARS can be defined as a string rewriting system (modulo a congruence) $\mathcal{GR} = \langle GR, \to \rangle$, where $GR = \{0, 1\}^*_{/\sim}$ and $\sim$ is defined as the transitive and reflexive closure of the relation defined by the following rule: $w \sim z$ if $w = x100y$ and $z = x011y$.

The rewrite relation is simply defined by the following two rules: $[w]_\sim \to [w1]_\sim$ and $[w]_\sim \to [w0]_\sim$. A detailed analysis of this ARS is postponed to the Appendix.

We now state some basic properties about sinks.

FACT 3.1.

1. *Every* $Sink(a)$ *is closed under* $\to$, *i.e.*, $y \in Sink(a)$ *and* $y \to y'$ *implies that* $y' \in Sink(a)$.
2. $Sink(a) \cap Sink(b) \neq \emptyset$ *if and only if* $a \downarrow b$.
3. *If* a *is a normal form, then* $Sink(a)$ *is CR.*

Proof.

1. Since $y \in Sink(a)$, we have $y \downarrow a$ and $\forall z.z \downarrow y \Rightarrow z \downarrow a$. Now $y' \downarrow y$, so $y' \downarrow a$. Further, if $z \downarrow y'$ then $z \downarrow y$ and hence $z \downarrow a$.

2. ($\Rightarrow$) Let $x \in Sink(a) \cap Sink(b)$. By definition of sink, $x \in Sink(a) \Rightarrow x \downarrow a$ and $x \downarrow y \Rightarrow a \downarrow y$. Since $x \in Sink(b)$, $x \downarrow b$, and hence $a \downarrow b$.

($\Leftarrow$) If $a \downarrow b$ then $\mathcal{G}(a) \cap \mathcal{G}(b) \neq \emptyset$, but by (1) for all x, $\mathcal{G}(x) \subseteq Sink(x)$.

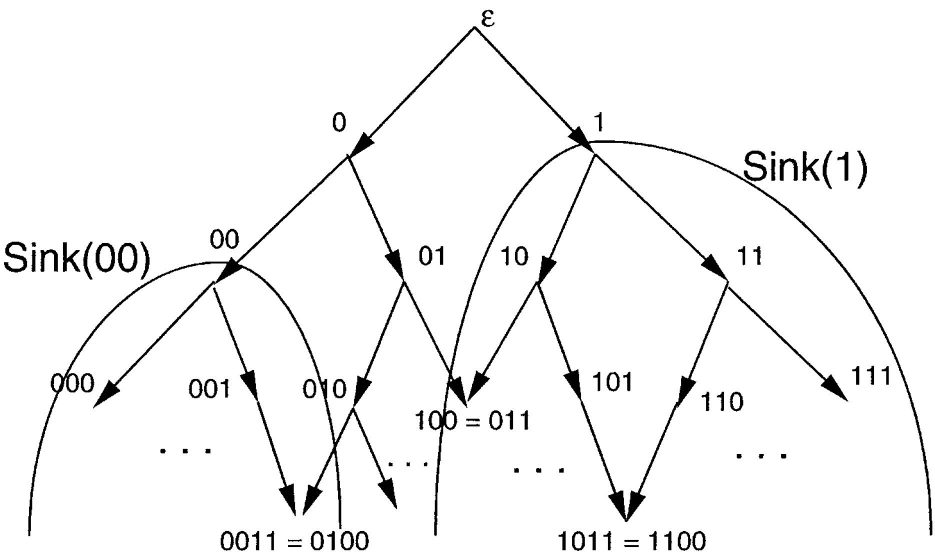

FIG. 6. The ARS described in Example 3.2.

3. If a is a normal form, then for all $x \in Sink(a)$ we have $x \xrightarrow{*} a$. As a consequence, $Sink(a)$ is confluent. ■

The following proposition is the main technical result and will be often referred to in the sequel:

LEMMA 3.1 (BOLT LEMMA). *Let $\mathcal{G}(a)$ be a WCR reduction graph. Then for each element $b \in \mathcal{G}(a)$ such that $Sink(b) \neq \mathcal{G}(a)$, there exists an infinite derivation $\mathcal{D} = a_0 \to a_1 \to \cdots \to a_n \to \cdots$ such that*:

1. $\forall i . a_i \in \mathcal{G}(a) \backslash Sink(b)$, *and*
2. $\forall n \in \mathbf{N} . \exists m \geq n . a_m \to Sink(b)$.

In the sequel we call such derivation a bolt, because of the typical shape that such derivations traditionally have in pictures.

Proof. Let $b \in \mathcal{G}(a)$ such that $Sink(b) \neq \mathcal{G}(a)$. Then there exists at least one element $x \in \mathcal{G}(a)$ such that $b \not\downarrow x$.

Then we can construct the following derivation: $a \equiv a_0 \xrightarrow{*} a_n \to b_1 \in Sink(b)$ with $a_n \notin Sink(b)$ (possibly $n = 0$). It is clear that $a_n \downarrow b$. So the only reason why $a_n \notin Sink(b)$ is that $a_n \downarrow c$, for some $c \not\downarrow b$. Therefore, there exists a derivation $\mathcal{D}_c = a_n \to a_{n+1} \to \cdots \to c'$ with $c' \in Sink(c)$. The element a_{n+1} has the following properties (see Fig. 7):

1. $a_{n+1} \notin Sink(b)$, because $a_{n+1} \downarrow c$.
2. $a_{n+1} \notin Sink(c)$, because otherwise the WCR property would be violated: By Fact 3.1(1), $a_{n+1} \in Sink(c) \Rightarrow \mathcal{G}(a_{n+1}) \subseteq Sink(c)$ and furthermore $\mathcal{G}(b_1) \subseteq Sink(b)$ and by Fact 3.1(2) we have $\mathcal{G}(a_{n+1}) \cap \mathcal{G}(b_1) = \emptyset$, but by the WCR property a_{n+1} and b_1 must have a common reduct.
3. $a_{n+1} \not\equiv a_n$.

Since we have $a_{n+1} \leftarrow a_n \to b_1$, by WCR there exists b_2 such that $a_{n+1} \xrightarrow{*} a_m \xrightarrow{*} b_2 \xleftarrow{*} b_1$, where a_m is the last element of the derivation not in $Sink(b)$ (possibly $a_m \equiv a_{n+1}$). Note that b_2 must be in $Sink(b)$, since, again by Fact 3.1(1), $Sink(b)$ is closed under $\to$. The same argument used for a_n works for a_m and so we can apply the above construction again to obtain the infinite derivation. Figure 7 graphically shows the basic idea of this proof. ■

Remark 3.1. We observe that, if $\mathcal{G}(a)$ is a SWCR reduction graph, there exist at least two elements satisfying the hypothesis of the above lemma. Moreover, because of the construction of the proof, $\mathcal{D}$ contains at least two distinct elements.

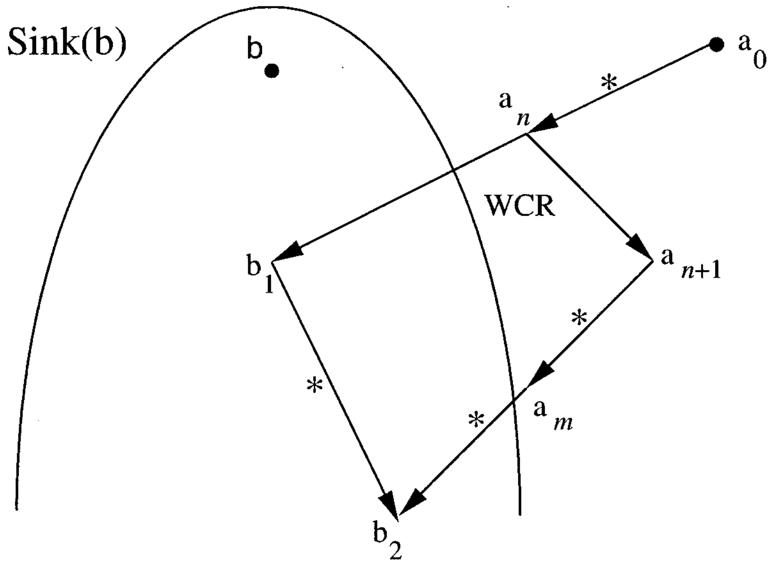

FIG. 7. Basic idea of the proof of the Bolt Lemma.

Figure 8 shows the general structure of a SWCR reduction graph. The following definition formalizes the intuition given by Fig. 8 in order to obtain the promised characterization of all SWCR ARS.

DEFINITION 3.2. A reduction graph $\mathcal{G}(a)$ is *bolt shaped* if there exist a bolt $\mathcal{D} : a_0 \to a_1 \to \cdots$ and elements $b, c_1, c_2, \ldots$ such that:

1. $a \equiv a_0$.
2. $\forall n \in \mathbf{N}. a_{2n+1} \to \mathit{Sink}(b)$.
3. $\forall n \in \mathbf{N}^+. a_{2n} \to \mathit{Sink}(c_n)$.
4. $\forall n \in \mathbf{N}. \mathit{Sink}(b) \cap \mathit{Sink}(c_n) = \emptyset$.

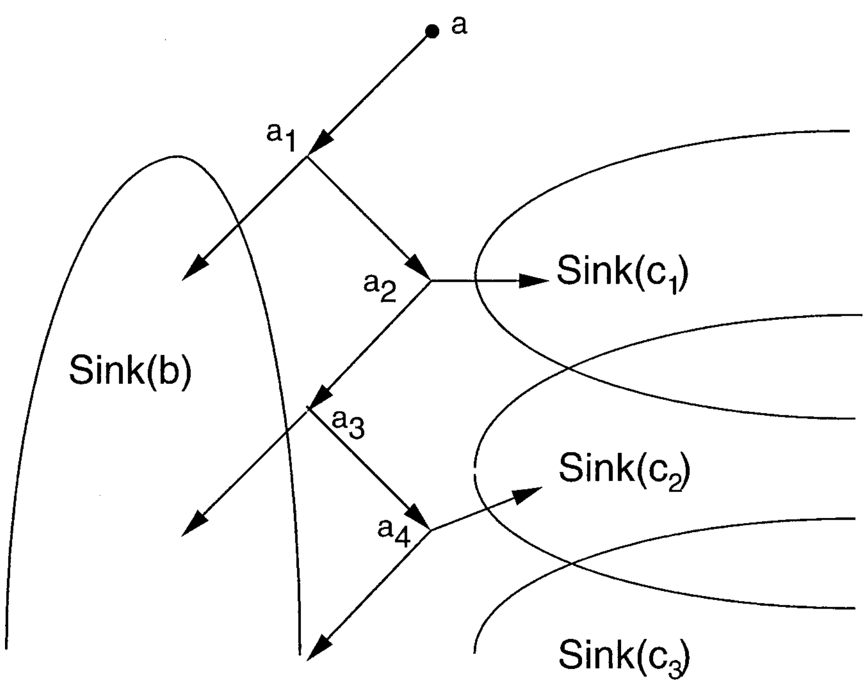

FIG. 8. General shape of an SWCR ARS.

5. $\forall n \in \mathbf{N}. a_n \notin Sink(b) \cup \bigcup_{k\in\mathbf{N}} Sink(c_k)$.
6. $\mathcal{G}(a) = \mathcal{D} \cup Sink(b) \cup \bigcup_{n\in\mathbf{N}} Sink(c_n)$.

We remark that we do not require that $Sink(c_i) \neq Sink(c_j)$, for $i \neq j \in \mathbf{N}$. We call $\mathcal{BS}$ the class of all bolt-shaped reduction graphs.

THEOREM 3.2. *Let $\mathcal{A}$ be a SWCR ARS. Then there is a bolt-shaped reduction graph embeddable into $\mathcal{A}$.*

Proof. If $\mathcal{A}$ is a SWCR ARS, then there exists at least one element a such that $\mathcal{G}(a)$ is not CR, so the statement easily follows by the Bolt Lemma and Remark 3.1. ■

4. FINITE DIMENSIONAL ARS

In this section, we identify a class of SWCR ARS, in which we can always embed one of the ARS of the family $\mathcal{F}$ defined above (see Remark 2.1 and Figs. 1 and 2).

To achieve this, it suffices to prove that there is a bolt between *two* sinks and that infinitely many elements of the bolt have a reduct in a unique derivation in each sink.

We now define three classes of WCR reduction graphs:

DEFINITION 4.1. Let $\mathcal{A} = (A, \rightarrow)$ be an ARS. We say that $I \subseteq A$ ($|I| \geq 2$) is a *family of incompatible elements* if $\forall x, y \in I. x \downarrow y \Rightarrow x \equiv y$ (i.e., all elements are pairwise incompatible).

We say that I is maximal if $\bigcup_{x\in I} Comp(x) = A$ (or equivalently $\forall x \in A. \exists y \in I. x \downarrow y$).

DEFINITION 4.2. Let $\mathcal{G}(a)$ be a WCR reduction graph.

1. $\mathcal{G}(a)$ is *weakly finite dimensional* (WFD) if there exists either a finite maximal family of incompatible elements, or it is CR.
2. $\mathcal{G}(a)$ is *iteratively finite dimensional* (IFD) if for all $x \in \mathcal{G}(a)$, $\mathcal{G}(x)$ is weakly finite dimensional.
3. $\mathcal{G}(a)$ is *strongly finite dimensional* (SFD) if either every family of incompatible elements is finite or it is CR.
4. If $\mathcal{G}(a)$ is WFD, we call $min_{I\in\mathcal{I}}|I|$ the *dimension* of $\mathcal{G}(a)$, where $\mathcal{I}$ is the set of all maximal families of incompatible elements. The dimension of a CR ARS is 1.

The notion of a finite dimensional WCR reduction graph is a notion of "regularity" of the reduction relation. For example, an SFD reduction graph is a sort of quasi-CR ARS. In some sense, this notion of dimension is close to the intuition of *how CR* a reduction graph is.

Remark 4.1. The condition $|I| \geq 2$ in Definition 4.1 is necessary; otherwise the definition of finite dimensional reduction graphs makes no sense. In fact, if we do not assume such a condition, all reduction graphs $\mathcal{G}(a)$ would have the set $\{a\}$ as a maximal family of incompatible elements, and hence each reduction graph would have dimension 1. This unfortunately forces us to introduce a special treatment for CR ARS in Definition 4.2(1), (2), and (4).

EXAMPLE 4.1. All reduction graphs in the examples considered up to now are at least WFD. Figure 9 shows the not WFD reduction graph $\mathcal{W}$. We postpone the analysis of this reduction graph to the Appendix.

Remark 4.2. Obviously, we have SFD $\Rightarrow$ IFD $\Rightarrow$ WFD. Now we show that the reverse implications do not hold, by giving appropriate counterexamples.

IFD $\not\Rightarrow$ SFD The reduction graph $\mathcal{GR}$ described in Example 3.2 is IFD, but not SFD, as shown in the Appendix.

WFD $\not\Rightarrow$ IFD Consider the reduction graph $\mathcal{N}'$ obtained by replacing, in the reduction graph $\mathcal{N}$ in Fig. 2, b with the reduction graph $\mathcal{W}$ that is not WFD. It is easy to see that $\{b', c\}$, where $\mathcal{G}(b') = \mathcal{W}$, is still a maximal family of incompatible elements in $\mathcal{N}'$, but $\mathcal{G}(b')$ is not WFD.

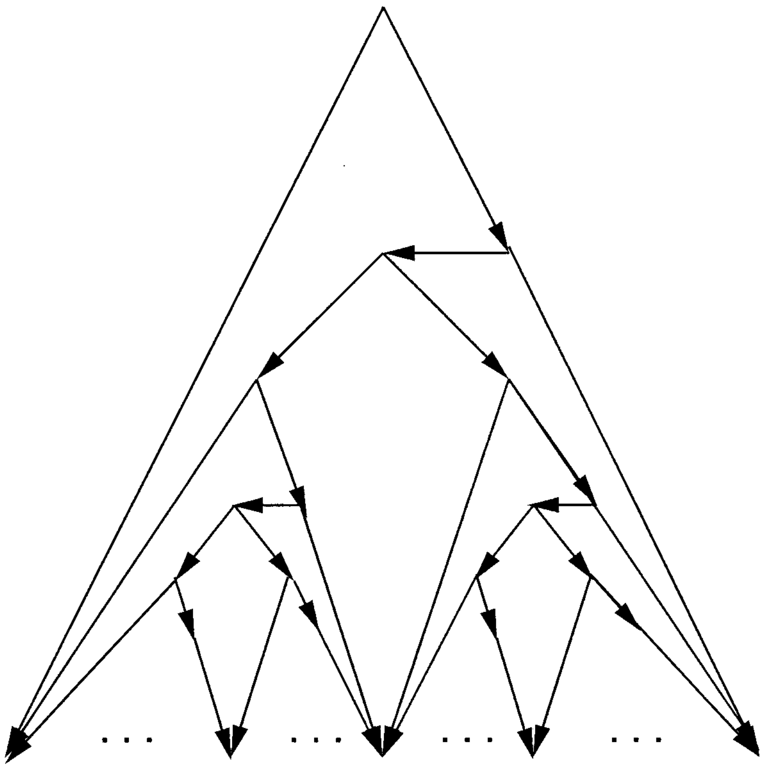

FIG. 9. $\mathcal{W}$, a not WFD reduction graph.

Under the hypothesis that $\mathcal{G}(a)$ is a weakly finite dimensional reduction graph, we can strengthen Lemma 3.1, showing that there exists a bolt that borders exactly *two* disjoint sinks.

PROPOSITION 4.1. *Let $\mathcal{G}(a)$ be a WFD SWCR reduction graph. Then there exist two distinct elements $c, b \in \mathcal{G}(a)$ such that $(c \not\downarrow b)$ and an infinite derivation $\mathcal{D}$: $a_0 \to a_1 \to \cdots \to a_n \to \cdots$ such that*:

1. $\forall i \in \mathbf{N}. a_i \in \mathcal{G}(a) \setminus (Sink(b) \cup Sink(c))$.
2. $\forall n \in \mathbf{N}. \exists m', m'' \geq n$

$$a_{m'} \overset{*}{\to} Sink(b)$$

$$a_{m''} \overset{*}{\to} Sink(c).$$

Proof. Let $I = \{c_1, \ldots, c_n\}$ be a maximal family of incompatible elements. Recall that, by definition, $n > 1$ (see Definition 4.1). Such an I does exist because, by the hypothesis that $\mathcal{G}(a)$ is SWCR, there are at least two incompatible elements in it. Without loss of generality, take $c_1 \in I$. Since c_1 has incompatible elements in $\mathcal{G}(a)$, $Sink(c_1) \neq \mathcal{G}(a)$, and by the Bolt Lemma there exists an infinite derivation $\mathcal{D}$: $a_0 \to a_1 \to \cdots \to a_n \to \cdots$, in which each element a_i is compatible both with c_1 and with some other element c_k, for $k \neq 1$, because $a_i \notin Sink(c_1)$. By the compatibility with c_1, for all $i \in \mathbf{N}$, $1 \leq k \leq n$, we have $a_i \notin Sink(c_k)$. Moreover, since I is maximal, each element in $\mathcal{D}$ is compatible with some element in I.

Hence, from each a_i starts at least one derivation that goes into some $Sink(c_k)(k > 1)$. Since I is a finite set, there exists at least one $c_{k_0}(k_0 > 1)$ such that the set

$$\left\{ n \in \mathbf{N} \,\middle|\, a_n \overset{*}{\to} Sink\left(c_{k_0}\right) \right\}$$

is an infinite set. The statement of the proposition is proved taking c_1 and c_{k_0} as b and c. ∎

Remark 4.3. The assumption that $\mathcal{G}(a)$ is a WFD reduction graph is necessary. As we will see in the Appendix, there exists no bolt between two sinks in the reduction graph $\mathcal{W}$.

One immediate consequence of Proposition 4.1 is the following:

COROLLARY 4.1. *Let $\mathcal{A}$ be a finite SWCR ARS. Then the Hindley ARS $\mathcal{H}$ is embeddable into $\mathcal{A}$.*

Proof. Since $\mathcal{A}$ is not CR, there is an element $a \in A$, such that $\mathcal{G}(a)$ is not CR. Since $\mathcal{G}(a) \subseteq \mathcal{A}$ is a finite reduction graph, it is obviously finite dimensional and hence, by Proposition 4.1, there is a bolt $\mathcal{D}$ between the sets of strongly compatible elements of two elements, say b and c. Since $\mathcal{D}$ is an infinite derivation and $\mathcal{G}(a)$ is a finite set, there are some elements that are repeated infinitely many times in $\mathcal{D}$. Let d be one of such elements and k the index of its first occurrence in $\mathcal{D}$. Let a_i and a_j be two distinct elements (recall Remark 3.1) such that

$$a_i \xrightarrow{*} b' \in Sink(b)$$

$$a_j \xrightarrow{*} c' \in Sink(c)$$

for some $b \not\downarrow c$ in $\mathcal{G}(a)$. Again, by Proposition 4.1, we can choose a_i, a_j such that $i, j \geq k$. Since we have infinite occurrences of d, we can take an index l such that $a_l \equiv d$ and $l \geq i, j$. Therefore, a_i and a_j belong to a cycle and we can define h as follows (see Fig. 1 for the labeling of nodes in $\mathcal{H}$):

$$h(\mathbf{a}) = a_i; \quad h(\mathbf{b}) = a_j; \quad h(\mathbf{c}) = c'; \quad h(\mathbf{d}) = b'.$$ ■

We now prove that in every SFD reduction graph a reduction graph in $\mathcal{F}$ is embeddable. The key property we use is stated in the following lemma, which says that an infinite set of elements in an SFD reduction graphs has always an infinite subset, such that each element in it has a reduct in a common derivation.

LEMMA 4.1. *Let $\mathcal{G}(a)$ be an SFD SWCR reduction graph. Let $X \subseteq \mathcal{G}(a)$ be an infinite set. Then there exists a derivation $\mathcal{D}$: $a_0 \to \cdots \to a_n \to \cdots$ such that*

$$\exists Y \subseteq X, Y \textit{ infinite set}, \forall y \in Y. \exists i \in \mathbf{N}. y \xrightarrow{*} a_i.$$

Proof. Let $X \subseteq \mathcal{G}(a)$ be an infinite set. We define a sequence of sets $\{X_n\}_{n \in \mathbf{N}}$ such that for all $n \in \mathbf{N}$, for all $x \in X_n$, x is a reduct of an element in X. Let $X_0 = X$. Since $\mathcal{G}(a)$ is an SFD reduction graph, there is at least an element $a_0 \in X_0$ such that $Comp(a_0) \cap X_0$ is an infinite set. We take $Y_0 = \{x \in X \mid x \xrightarrow{*} a_0\}$ as the first approximation of Y. If Y_0 is an infinite set we are done since the derivation a_0 and the set Y_0 satisfy the statement. Otherwise, let $Z_0 = (X_0 \cap Comp(a_0)) \backslash Y_0$. Now we define

$$X_1 = \{\forall y \in Z_0 \text{ choose an } x \text{ such that } a_0 \xrightarrow{*} x \text{ and } y \xrightarrow{*} x\}.$$

If at least one element $a_1 \in X_1$ is a reduct of an infinite number of elements in X_0 we are done, because the derivation $\mathcal{D}$: $a_0 \xrightarrow{*} a_1$ satisfies the statement (this always happens when X_1 is a finite set). Otherwise, the same argument applied for X_0 works now for X_1, so we can choose an element $a_1 \in X_1$ such that $Comp(a_1) \cap X_1$ is an infinite set. We take $Y_1 = Y_0 \cup \{x \in X \mid \exists y \in X_0 . x \xrightarrow{*} y \xrightarrow{*} a_1\}$. Observe that $Y_0 \subset Y_1$, since we remove from Z_0 elements that are in Y_0. Taking $Z_1 = (X_1 \cap Comp(a_1)) \backslash \{y \in X_1 \mid y \xrightarrow{*} a_1\}$, we define

$$X_2 = \{\forall y \in Z_1 \text{ choose an } x \text{ such that } a_1 \xrightarrow{*} x \text{ and } y \xrightarrow{*} x\}$$

and so on. Having chosen in this way elements $a_0 \xrightarrow{*} a_1 \xrightarrow{*} \cdots \xrightarrow{*} a_k$, let $(Z_k = X_k \cap Comp(a_k) \backslash \{y \in X_k \mid y \xrightarrow{*} a_k\})$:

$$X_{k+1} = \{\forall y \in Z_k \text{ choose an } x \text{ such that } a_k \xrightarrow{*} x \text{ and } y \xrightarrow{*} x\}.$$

If, for some $k \in \mathbf{N}$, in X_{k+1} there is an element that is a reduct of an infinite number of elements in X_k, such element is also a reduct of an infinite number of elements in X and hence we obtain a finite derivation. Otherwise we obtain an infinite derivation, where for every k the element a_{k+1} is a reduct of elements in X that have not $a_0, \ldots a_k$ as reducts, so that the set

$$Y_{k+1} = \{x \in X \mid \exists y \in X_k . x \xrightarrow{*} y \xrightarrow{*} a_{k+1}\}$$

strictly extends Y_k. Finally take $Y = \bigcup_{k \in \mathbf{N}} Y_k$. ■

THEOREM 4.1. *Let $\mathcal{G}(a)$ be a SFD SWCR reduction graph. Then we can embed in it at least one of the reduction graphs in $\mathcal{F}$.*

Proof. By Proposition 4.1 there is a bolt $\mathcal{D}$ between two sinks, say $Sink(a)$ and $Sink(b)$. Now we define the set

$$X_a \triangleq \{x \in Sink(a) \mid \exists y \in \mathcal{D}. y \xrightarrow{*} x\}.$$

Similarly we define X_b. If a finite number of distinct elements appear in the bolt $\mathcal{D}$, then $\mathcal{D}$ contains a cycle, and reasoning as in the proof of Corollary 4.1, we can show that $\mathcal{H}$ is embeddable in $\mathcal{G}(a)$. If $\mathcal{D}$ contains an infinite number of distinct elements, then there is an infinite number of distinct arrows from $\mathcal{D}$ into X_a and X_b:

1. X_a is a finite set. Then there is an element $x_a \in X_a$ that receives an infinite number of arrows from $\mathcal{D}$.
2. X_a is an infinite set. By Lemma 4.1, there is a derivation $\mathcal{D}'$ in $Sink(a)$ in which there are the reducts of an infinite number of elements in X_a. Here we can distinguish again two cases:
   (i) $\mathcal{D}'$ is a finite derivation.
   (ii) $\mathcal{D}'$ is an infinite derivation.

If both X_a and X_b are in situation 1 or 2 (i) we can embed in $\mathcal{G}(a)$ the reduction graph $\mathcal{N}$, whereas if they are both in situation 2 (ii) we can embed in $\mathcal{G}(a)$ the reduction graph $\mathcal{K}$. Otherwise we can embed in $\mathcal{G}(a)$ the reduction graph $\mathcal{O}$. ■

An IFD, not SFD, ARS

We now present a string rewriting system (Fig. 10) that is IFD, but not SFD, and we show that we cannot embed in it any reduction graph in $\mathcal{F}$.

EXAMPLE 4.2. We define the rewrite system, $\mathcal{A} = \langle A, \rightarrow \rangle$. $A = \{a, b\}^* \cup \{a, b\}^* \cdot c \cup \mathbf{N}$. The rewriting relation is defined by the following rules, which make use of a function $n \colon \{a, b\}^* \rightarrow \mathbf{N}$ defined below ($w \in \{a, b\}^*$):

$$w \rightarrow n(w)$$
$$w \rightarrow wc$$
$$wc \rightarrow wb$$
$$wc \rightarrow wa.$$

In the following we call a reduction step $wc \rightarrow wa$ (resp. $wc \rightarrow wb$) an a-step (resp. a b-step). The function n is recursively defined as

$$n(\varepsilon) = 0$$
$$n(wb) = n(w) + 1 \quad w \in \{\varepsilon\} \cup \{a, b\}^* \cdot b$$
$$n(wa^k b) = n(w) + k \quad w \in \{\varepsilon\} \cup \{a, b\}^* \cdot b$$
$$n(wa) = n(w) \quad w \in \{a, b\}^*.$$

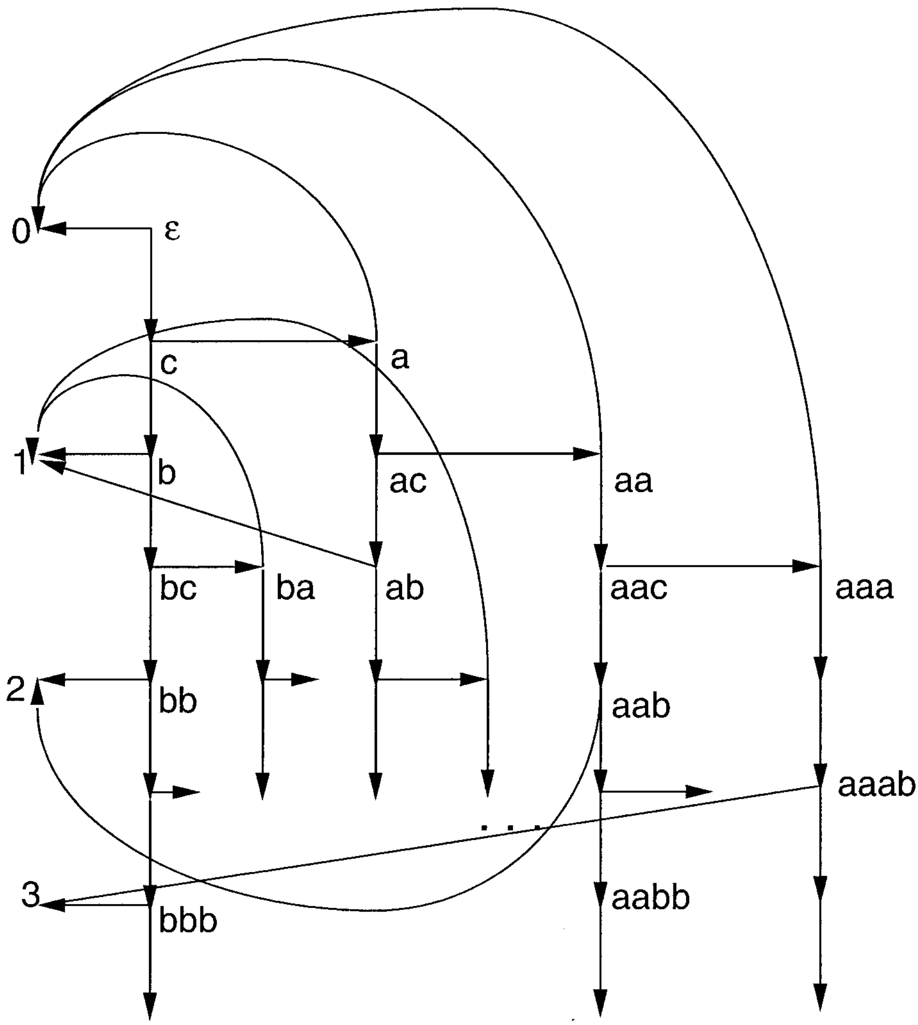

FIG. 10. Some intuition about the structure of the ARS in Example 4.2.

FACT 4.2. *Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be the reduction graph defined above. Then we have*:

1. *The set of normal forms is* **N**.
2. *For every string $w \in A \backslash \mathbf{N}$ there is only one string w' such that $w' \rightarrow w$.*
3. *All strings in $A \backslash \mathbf{N}$ are pairwise compatible.*

Proof.

1. This is proven by observing that the patterns on left-hand sides of rewriting rules match with all elements in A, except for those in **N.**

2. This is proven by definition of $\rightarrow$. If $w = w'a$ (resp. $w'b$) we can reach it only with an a-step (resp. a b-step) from $w'c$. Moreover, the only in-going arrow of $w'c$ comes from w'.

3. For every element $w \in A$, there exists $m_w \in \mathbf{N}$, such that $\{n \in \mathbf{N} \mid n > m_w\} \subset \mathcal{G}(w)$ (consider the derivation starting in w without a-steps). Hence two elements, $w, z \in A \backslash \mathbf{N}$, have as common reducts the set of normal forms $\{n \in \mathbf{N} \mid n > max(m_w, m_z)\}$. ■

Observe that (2) implies that we can close a peak only on normal forms (if we remove normal forms and its in-going arrows, we obtain a tree), whereas (3) implies that each family of incompatible elements contains at most one element in $\mathcal{A} \backslash \mathbf{N}$.

PROPOSITION 4.2. *$\mathcal{A}$ is WCR.*

Proof. There are two kinds of peaks:

1. $n(w) \leftarrow w \rightarrow wc$. We have that $wc \rightarrow wa \rightarrow n(w)$, by definition of n.
2. $wa \leftarrow wc \rightarrow wb$. Let $w = w'a^k b$ and $w' \in \{\varepsilon\} \cup \{a, b\}^* \cdot b$. Again by definition of n:
   (i) $k = 0$. We have $wa \rightarrow wac \rightarrow wab \rightarrow n(w) + 1 \leftarrow wb$.
   (ii) $k > 0$. We have $w'a^{k+1} \rightarrow w'a^{k+1}c \rightarrow w'a^{k+1}b \rightarrow n(w) + k + 1 \leftarrow w'a^k bb \leftarrow w'a^k bc \leftarrow w'a^k b$. ■

FACT 4.3. *$\mathcal{A}$ is an IFD, not SFD, reduction graph.*

Proof. For every $w = w'b$, $\{n(w), wb\}$ is a maximal family of incompatible elements in $\mathcal{G}(w)$, because the set of normal forms of $\mathcal{G}(wb)$ contains all normal forms of $\mathcal{G}(w)$ except $n(w)$. Similarly, we can find a finite maximal family of incompatible elements when $w = w'a$. Therefore $\mathcal{A}$ is an IFD reduction graph. It is not SFD, because **N**, as an example, is an infinite maximal family of incompatible elements. ■

PROPOSITION 4.3. *No reduction graph in $\mathcal{F}$ is embeddable in $\mathcal{A}$.*

Proof. By Fact 4.2(2), the only possible reduction graph in $\mathcal{F}$ that we can embed in $\mathcal{A}$ is $\mathcal{N}$.
Hence we should find two normal forms and an infinite derivation that keeps compatibility with both of them. However such derivation cannot exist, because:

1. Since, trivially, $n(w) \geq |w|_b$, the derivation cannot contain infinitely many b-steps, because such a derivation loses compatibility with any normal form after a finite number of steps.
2. The infinite derivation starting in w, which does not contain b-steps, keeps compatibility with $n(w)$, but, for every $k \in \mathbf{N}$, after k a-steps loses compatibility with all normal forms less than $n(w) + k$ and different from $n(w)$. ■

5. APPLICATIONS

In this section, we state some results (or new proofs for known results) that can be easily obtained by means of the technical tools introduced in this work, in particular from the Bolt Lemma.

Boundaries between CR and WCR ARS

It is well known that WCR & SN $\Rightarrow$ CR (Newman's lemma). An elegant proof by *noetherian induction* is given in [6].
The following results show other "border conditions" that force a WCR ARS to be CR.

LEMMA 5.1. *Let $\mathcal{G}(a)$ be an SWCR reduction graph. If $\mathcal{G}(a)$ is SN^{-1} and FB^{-1} then there is no normal form in $\mathcal{G}(a)$.*

Proof. This is proven by contradiction. Suppose that there exists a normal form $b \in \mathcal{G}(a)$. The element b has incompatible elements in $\mathcal{G}(a)$; otherwise $Sink(b) = \mathcal{G}(a)$ and, by Fact 3.1(3), $\mathcal{G}(a)$ would be CR.
By the Bolt Lemma, there exists an infinite derivation $\mathcal{D}$: $a_0 \rightarrow a_1 \rightarrow \cdots$, such that $a_k \rightarrow Sink(b)$, for infinitely many indexes $k \in \mathbf{N}$. Since SN^{-1} implies that $\mathcal{G}(a)$ is acyclic, all such elements are distinct and hence there is an infinite number of distinct in-going arrows in $Sink(b)$.
Now we consider the set $J \triangleq \{x \in Sink(b) \mid \exists y \in \mathcal{D}. y \rightarrow x\}$. We can have two situations:

- J is finite, then at least one of its elements receives infinitely many arrows, so the reduction graph $\mathcal{G}(a)$ is not FB^{-1}.
- J is infinite. In this case, consider the ARS $\mathcal{A}' = \langle A', \leftarrow \rangle$, where $A' \triangleq \{x \mid x \stackrel{*}{\rightarrow} b\}$. Since $\mathcal{G}(a)$ is acyclic and $J \subseteq A'$, $\mathcal{A}'$ is an infinite DAG. By the König's lemma, $\mathcal{A}'$ has either an infinite derivation, which implies that $\mathcal{G}(a)$ is not SN^{-1}, or some nodes with infinitely many one-step reducts, which implies that $\mathcal{G}(a)$ is not FB^{-1}. ■

THEOREM 5.1. *Let $\mathcal{A} = \langle A, \rightarrow\rangle$ be a WCR ARS. If $\rightarrow$ is SN^{-1} and FB^{-1}, then $\mathcal{A}$ has the UN property.*

Proof. If $\mathcal{A}$ is CR, then $\mathcal{A}$ has the UN property. Let us suppose that $\mathcal{A}$ is SWCR. Toward a contradiction, we suppose that there are two normal forms $a, b \in A$ such that $a = b$ and $a \not\equiv b$. This implies, in particular, that there exists an element c such that $c \xrightarrow{*} a$ and $\mathcal{G}(c)$ is not CR. By Lemma 5.1, $\mathcal{G}(c)$, and hence $\mathcal{A}$, is either not SN^{-1} or not FB^{-1} against the hypothesis. ■

Using known properties of ARS (namely FB^{-1} & $SN^{-1} \Rightarrow$ Inc, UN & WN $\Rightarrow$ Ind and Inc & Ind $\Rightarrow$ SN & CR, see [8], Fig. 2.2) the following already known result easily follows as corollary:

COROLLARY 5.1. *Let $\mathcal{A} = \langle A, \rightarrow\rangle$ be a WCR ARS. If $\mathcal{A}$ is SN^{-1}, FB^{-1} and WN then $\rightarrow$ is CR and SN.*

An Extension of the Newman's Lemma

DEFINITION 5.1. Let $\mathcal{A}$ be an ARS. We say that $\mathcal{A}$ is *eventually Church–Rosser* (ECR), if every infinite derivation goes into a CR sub-ARS after a finite number of steps.

THEOREM 5.2. *Let $\mathcal{A} = \langle A, \rightarrow\rangle$ be an ARS. $\mathcal{A}$ is CR if and only if $\mathcal{A}$ is ECR and WCR.*

Proof. ($\Rightarrow$) Trivial.
($\Leftarrow$) Let $\mathcal{A}$ be a SWCR and ECR ARS. So there exists $a \in A$ such that $a \xrightarrow{*} b$ and $a \xrightarrow{*} c$, with $b \not\downarrow c$. By the Bolt Lemma, there exists an infinite derivation $\mathcal{D}$ such that $\forall x \in \mathcal{D}.\exists y. y \not\downarrow b \;\&\; x \downarrow b \;\&\; x \downarrow y$, but this contradicts the hypothesis that $\mathcal{A}$ is an ECR ARS. ■

Since SN $\Rightarrow$ ECR, but obviously ECR $\not\Rightarrow$ SN, the above theorem strictly extends Newman's lemma and the above proof can be seen also as an alternative proof for it. Moreover, the above theorem gives an alternative characterization of the CR property. However, we do not know whether it can be used to show the CR property, instead of Newman's lemma, when termination cannot be required. In particular, it is not clear whether there exist some concrete structures where ECR property is easier to prove than CR.

More on Finite SWCR ARS

Finally, we apply our results to state a nice combinatorial property of finite WCR reduction graphs.

We define the concept of Hindley Point; intuitively an element of an ARS is a Hindley Point if it belongs to a cycle and it has a one-step reduct from which we cannot come back to the cycle. Moreover, we require that from such a cycle we can go into at least two incompatible elements. In WCR ARS this also implies that such a cycle consists of two or more steps. The next theorem establishes an inequality between the number of normal forms and the number of Hindley Points in finite SWCR reduction graphs.

DEFINITION 5.2. Let $\mathcal{A}$ be an ARS. We say that $x \in A$ is a *Hindley Point* if there is an embedding $h: \mathcal{H} \rightarrow \mathcal{A}$ such that $x = h(a)$. We call $HP_{\mathcal{A}} = \{x \in A \mid x \text{ is a Hindley Point}\}$ the set of Hindley Points.

THEOREM 5.3. *Let $\mathcal{G}(a)$ be a finite SWCR reduction graph and let $NF_{\mathcal{G}(a)}$ be the set of normal forms of $\mathcal{G}(a)$. Then the following inequality holds*:

$$\left|NF_{\mathcal{G}(a)}\right| \leq \left|HP_{\mathcal{G}(a)}\right|.$$

Proof. Let c be a normal form. Since c belongs to a SWCR reduction graph, by the Bolt Lemma there is an infinite derivation $\mathcal{D}: a_0 \rightarrow \cdots \rightarrow a_n \rightarrow \cdots$ with all a_n compatible with c_1 and infinitely many indexes $i \in I$ in this derivation, such that a_i one-step reduces to an element in $Sink(c)$. Since $\mathcal{G}(a)$ is finite, at least one element a_c appears infinitely many times as a_i, $i \in I$, and hence belongs to a cycle. Furthermore a_c does not reduce one-step to any element of $Sink(b)$ for $b \not\downarrow c$, by WCR property. Hence, reasoning as in the proof of Theorem 4.1, we can construct an embedding $h: \mathcal{H} \rightarrow \mathcal{G}(a)$ with $h(a) = a_c$. Therefore a_c is an Hindley Point and by the observation above, $c \mapsto a_c$ is an injection of $NF_{\mathcal{G}(a)}$ into $HP_{\mathcal{G}(a)}$. ■

APPENDIX

A Detailed Analysis of the Reduction Graphs $\mathcal{GR}$ and $\mathcal{W}$

One interesting aspect of the reduction graphs $\mathcal{GR}$ and $\mathcal{W}$, introduced respectively in Examples 3.2 and 4.1, is that we can generate them in a constructive way by simple "graph-rewriting" rules (in this context our approach will be somewhat informal). These rules, R_{GR} and R_W (Figs. 11 and 14), are a little more complex than that showed in Fig. 12, which generates a CR ARS (by $*$ on the edges we mean that the edge has not to be added if it already exists).

The Reduction Graph $\mathcal{GR}$

Figure 13 shows the result of three levels of parallel applications of the generating rule R_{GR}. It is straightforward to see, by induction on the length of words (or simply by an adequate labeling of nodes), that the ARS generated is $\mathcal{GR}$. We must show that $\mathcal{GR}$ is not CR. In order to do this, we show that we can express the number of equivalence classes of words of length n by a Fibonacci-like recurrent relation. Let $GR_n \stackrel{\Delta}{=} |\{[w]_{/\sim} \mid \#w = n\}|$, where $\sim$ is the equivalence relation defined in Example 3.2. Then we have the following:

Proposition A.1.

$$
\begin{aligned}
GR_0 &= 1\\
GR_1 &= 2\\
GR_{n+2} &= GR_{n+1} + GR_n + 1.
\end{aligned}
$$

Proof. This is proven by induction on n. For $n \le 2$, the statement holds trivially. Now let us suppose that the statement holds for $n + 2$. We have

$$GR_{n+3} = 2GR_{n+2} - GR_n$$

since each element of length $n + 2$ has two descendants of length $n + 3$, and for all w such that $\#w = n$, we have $w100 \sim w011$. Hence, by inductive hypothesis, we have

$$
\begin{aligned}
GR_{n+3} &= 2GR_{n+1} + 2GR_n + 2 - GR_n\\
&= (GR_{n+1} + GR_n + 1) + GR_{n+1} + 1\\
&= GR_{n+2} + GR_{n+1} + 1.
\end{aligned}
$$ ■

Proposition A.2. *$\mathcal{GR}$ is not CR.*

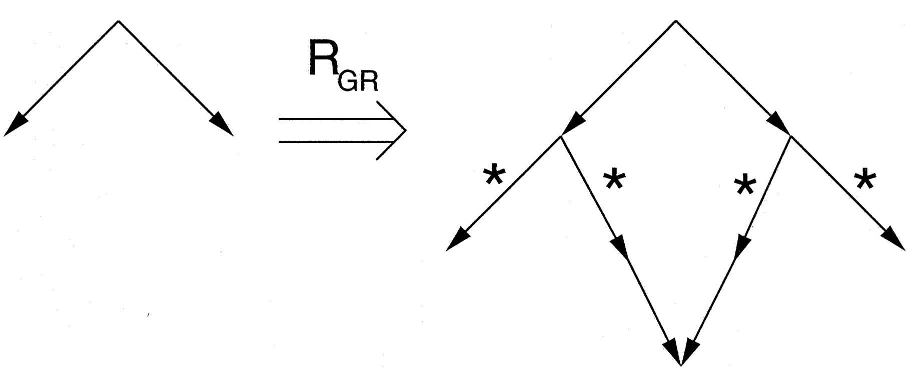

FIG. 11. A "graph-rewriting" rule for the ARS of Example 3.2.

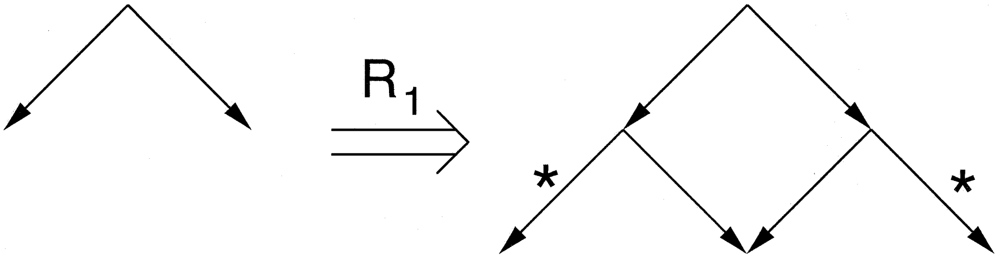

FIG. 12. A "graph-rewriting" rules that generates a CR ARS.

Proof. Let us consider the nodes labeled respectively by 00 and 1. At level $n+2$ the reducts of 1 are the GR_{n+1} right-most elements, whereas the reduct of 00 are the GR_n left-most ones. Since, by Proposition A.1, $GR_{n+2} = GR_{n+1} + GR_n + 1$, we have that for each n, $\Delta^{n+2}(\varepsilon) = \Delta^n(00) \cup \Delta^{n+1}(1) \cup \{[b_{n+2}]_{/\sim}\}$, where $b_{2n} = 01^n$. ■

By a similar reasoning, we can show that the number of applications of R_{GR} at level n (and accordingly the number of open peaks at level $n-1$) is GR_n. We observe that b_{n+2} is the $(n+2)$-th element in the bolt between *Sink*(00) and *Sink*(1) and that $b_{2n} = [(01)^n]_{/\sim}$. This immediately implies that the dimension of $\mathcal{GR}$ is 2. Since for each $w \in \{0, 1\}^*$ we have that $\mathcal{G}([w]_{/\sim})$ is isomorphic to $\mathcal{GR}$, $\mathcal{GR}$ is an IFD ARS. To see that $\mathcal{GR}$ is not SFD, we can consider the family of elements $\mathcal{I} = \{[00^{2n}1]_{/\sim} \mid n \in \mathbf{N}\}$.

The Golden Notation for Computable Real Numbers

An alternative interesting ("semantic") approach to show that $\mathcal{GR}$ is not CR is based on a strong similarity between $\mathcal{GR}$ and a binary representation for exact real numbers' computability that uses as basis the golden number $b = (1 + \sqrt{5})/2$ [3, 4].

DEFINITION A.1. We define the *interpretation* function $\|\cdot\|: GR \to \mathbf{R}$ as (where $w \in \{0, 1\}^*$)

$$\|w\| = \sum_{i=1}^{\#w} w_i b^{-i}.$$

By properties of $\sim$ and $=$, it is easy to prove the following:

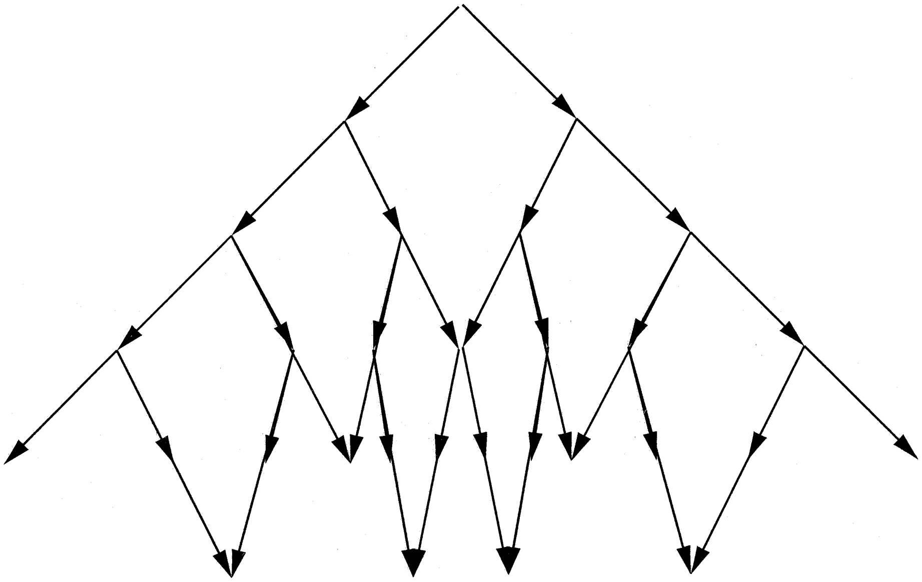

FIG. 13. The result of the three-level parallel applications of R_{GR}.

Proposition A.3.

$$w \sim z \Rightarrow \|w\| = \|z\|.$$

We can now give an alternative proof of Proposition A.2 that $\mathcal{GR}$ is not CR, showing that the interpretation of each reduct of 00 is strictly less than the interpretation of each reduct of 1 and hence $[00]_{/\sim} \not\downarrow [1]_{/\sim}$, by Proposition A.3.

In fact, the interpretation of each reduct of 00 is strictly less than $\|001^\omega\|$ and the interpretation of each reduct of 1 is greater or equal to $\|10^\omega\|$:

$$\|10^\omega\| = b^{-1}$$
$$\|001^\omega\| = \frac{1}{b^2(b-1)}.$$

Moreover, we observe that $b^{-1} = 1/b^2(b-1)$, by properties of the golden number. Let w be a common reduct of 00 and 1. It follows that

$$\|w\| < \frac{1}{b^2(b-1)} = b^{-1} \leq \|w\|,$$

which yields a contradiction.

The Reduction Graph $\mathcal{W}$

First of all, we note that $\mathcal{W}$ is a WCR reduction graph, simply observing that each "redex" contains a "peak" and that the application of R_W closes this peak (and generates two new redexes).

In order to show that $\mathcal{W}$ is not WFD, it is convenient to introduce a labeling of nodes (see Fig. 14). Each node is labeled by a closed interval (in Fig. 14 and in the following, we abbreviate $[a, a]$ with a and we omit to write labels of nodes that have the same label as their parent node). If we generate $\mathcal{W}$ starting from the left-hand side of R_W, with $a = 0$ and $b = 1$, normal forms have labels in the set $\{\frac{k}{2^n} \mid n \in \mathbf{N}, 0 \leq k \leq 2^n\}$. An element x, which is labeled by the interval $[l_x, r_x]$, has as reducts all normal forms labeled by an interval $[y, y]$, such that $l_x \leq y \leq r_x$. Since two distinct elements are compatible if and only if they have a normal form as a common reduct, for all $x, y, x \downarrow y$ if and only if $[l_x, r_x] \cap [l_y, r_y] \neq \emptyset$.

Proposition A.4. *$\mathcal{W}$ is not WFD.*

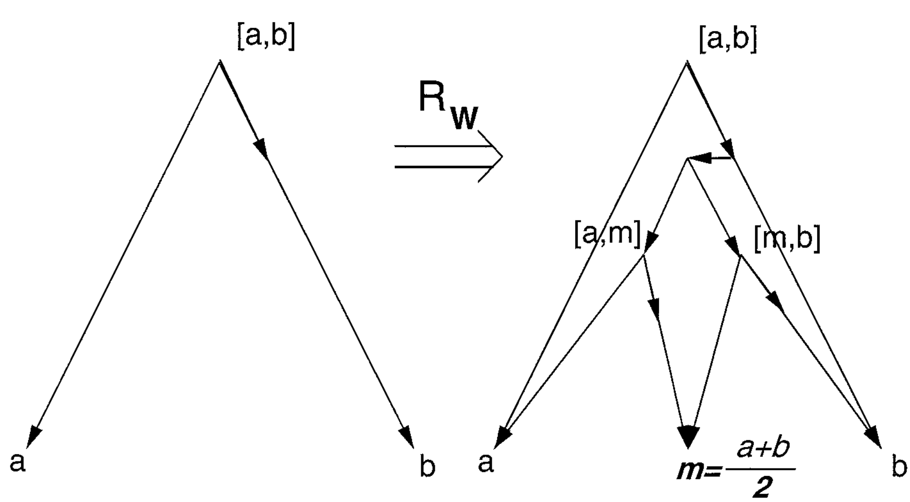

FIG. 14. The generating rule R_W for $\mathcal{W}$.

Proof. Suppose that there exists a finite maximal family of incompatible elements $I = \{c_1, \ldots, c_n\}$. Since $c_i \not\downarrow c_j$ for $i \neq j$, the intervals $[l_{c_i}, r_{c_i}]$ and $[l_{c_j}, r_{c_j}]$ are disjoint. So we can define a total order on I as

$$b \preceq c \Leftrightarrow r_b < l_c.$$

Let $b \preceq c$ be two elements in I. Then $l_c - r_b > 0$ and hence there exists a normal form x (actually infinitely many such normal forms) such that $l_c < l_x = r_x < r_b$. Hence x is not compatible with any element in I and therefore I is not maximal. ∎

PROPOSITION A.5. *For all* $a, b \in \mathcal{W}$, $a \not\downarrow b$, *there is no bolt between Sink*(a) *and Sink*(b).

Proof. Assume without loss of generality that $r_a < l_b$ (recall that $a \not\downarrow b$ implies that $[l_a, r_a] \cap [l_b, r_b] = \emptyset$). Suppose that there exists a bolt $\mathcal{D}: a_0 \to a_1 \to \cdots$ between *Sink*(a) and *Sink*(b). Since $\mathcal{D}$ must keep compatibility with both a and b, then for all $a_i \in \mathcal{D}$, $[l_{a_i}, r_{a_i}] \cap [l_a, r_a] \neq \emptyset$ and $[l_{a_i}, r_{a_i}] \cap [l_b, r_b] \neq \emptyset$. Observe that in Fig. 14, going through the node labeled with $[a, m]$ (resp. $[m, b]$) we lose compatibility with all normal forms labeled by a number in $(m, b]$ (resp. $[a, m)$). However, every infinite derivation goes through such nodes infinitely many times; otherwise the derivation would stop on a normal form after a finite number of steps. If $r_{a_n} - l_{a_n} = d$ and the derivation $a_n \xrightarrow{*} a_m$ goes trough k such nodes, $r_{a_m} - l_{a_m} = \frac{1}{2^k} d$. Therefore, there exists an m such that $r_{a_m} - l_{a_m} < l_b - r_a$ and a_m cannot be compatible with both a and b. ∎

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